RELAXATION OSCILLATIONS IN SYSTEMS WITH DIFFERENT TIME SCALES

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Roughly speaking, a relaxation oscillation (RO) is a periodic motion where at least one component is nearly a discontinuous periodic function. This property suggests to look for ROs as solutions of dynamical systems of the type

(1) $\frac{dx}{dt} = f(x,y,\varepsilon,\alpha), \qquad \frac{dy}{dt} = g(x,y,\varepsilon,\alpha)$,

where $x \in R^n$, $y \in R^m$, $\alpha \in R^p$, ε is a small parameter, f and g are sufficiently smooth.

A system of type (1) is called a singularly perturbed autonomous differential system when $g(x,y,0,\alpha)$ is not identically zero. It represents a differential system with at least two time-scales.

Examples of relaxation oscillations can be found in engineering as well as in biosciences [2,5]. Another motivating example is the existence of periodic travelling waves $u(t,x) = U(x-ct)$ in reaction-diffusion systems [12,13] of the form

(2) $\frac{\partial u}{\partial t} = \nu \triangle u + f(u,\lambda), \quad u \in R^n$,

where ν measures the diffusion, λ is some parameter, and c represents the velocity of propagation. This problem is equivalent to the question for periodic solutions of the differential equation

$$\nu \frac{d^2U}{dz^2} + c\frac{dU}{dz} + f(U,\lambda) = 0 .$$

Setting $z =: -c\tau$, $\varepsilon := \nu/c^2$ we obtain the system

(3) $\frac{dU}{d\tau} = V , \quad \varepsilon\frac{dV}{d\tau} = V - f(U,\lambda)$.

In case ε to be small (3) is a singularly perturbed system of type (1).

The system

(4) $\frac{dx}{dt} = f(x,y,0,\alpha), \qquad 0 = g(x,y,\varepsilon,\alpha)$,

is called the degenerate system to (1). It represents a differential algebraic system.

There are two basic approaches in establishing ROs in (1) depending on the property whether the system (4) defines a dynamical system on some manifold defined by $g(x,y,0,\alpha) = 0$ or not. The latter case was studied by Pontrjagin, Mishchenko and Rozov [8,9] for the following class of differential systems

$$(5) \qquad \frac{dx}{dt} = f(x,y), \qquad \varepsilon \frac{dy}{dt} = g(x,y) .$$

Introducing the concept of a discontinuous solution of the corresponding degenerate system and using the asymptotic expansion of a solution of (5) with respect to ε they prove a theorem on the existence of a unique stable (unstable) RO near a unique stable (unstable) discontinuous periodic solution Π_0 in case $n=m=1$. In higher dimensional cases they can prove only the existence of a RO near Π_0, there is no result on stability and uniqueness. In what follows we indicate how these results can be improved.
For $\varepsilon \neq 0$ we introduce the fast time τ by $t =: \varepsilon\tau$ and rewrite (1) in the form

$$(6) \qquad \frac{dx}{d\tau} = f(x,y,\varepsilon,\alpha), \qquad \frac{dy}{d\tau} = g(x,y,\varepsilon,\alpha) .$$

For $\varepsilon \neq 0$, the systems (1) and (3) have the same phase picture. Let us assume that (6) has a periodic solution for $0 < \varepsilon \leqq \varepsilon_0$, let $\Gamma_{\varepsilon,\alpha}$ be the corresponding orbit. We suppose $\Gamma_{\varepsilon,\alpha}$ to converge to a closed invariant curve $\Gamma_{0,\alpha}$ of (6) as $\varepsilon \to 0$. $\Gamma_{\varepsilon,\alpha}$ is said to be an intrinsic RO if $\Gamma_{\varepsilon,\alpha}$ is near $\Gamma_{0,\alpha}$ and if $\Gamma_{0,\alpha}$ contains at least two different continua of equilibria of (6) for $\varepsilon = 0$ and their connecting orbits.
The problem of existence of an intrinsic RO can be treated as a problem of persistence of some closed invariant curve of the system (6) for $\varepsilon = 0$ as ε varies. By this way, using results on the persistence of integral manifolds [6] the theorem of Mishchenko and Rozov in case $n=m=1$ can be extended to systems of type (1), at the same time the smoothness conditions on f and g may be relaxed. Details can be found in a forthcoming paper.
To be able to obtain a uniqueness and stability result for higher order systems we consider the case where the degenerate system (4) defines a dynamical system on some manifold $\mathcal{M}_\alpha := \{(x,y) \in R^{n+m} : y = \varphi(x,\alpha)\}$ where $g(x,\varphi(x,\alpha),0,\alpha) \equiv 0$. The corresponding dynamical system is called the reduced system to $\mathcal{M}_\alpha$

$$(7) \qquad \frac{dx}{dt} = f(x,\varphi(x,\alpha),0,\alpha) =: \tilde{f}(x,\alpha) .$$

It is well-known that under some conditions the existence of a periodic solution $x = p(t,\alpha)$ of the reduced system (7) implies the existence of a periodic solution $(\bar{x}_p(t,\varepsilon,\alpha),\bar{y}_p(t,\varepsilon,\alpha))$ of the full system

(1) for sufficiently small ε (Theorem of Anosov [1], theory of integral manifolds [3,7,11]). In this context we have to note that if $p(t,\alpha)$ is no RO then the same holds for $(\bar{x}_p(t,\varepsilon,\alpha),\bar{y}_p(t,\varepsilon,\alpha))$. Thus, in this situation the parameter ε can be characterized only as a continuation parameter, not as parameter generating a RO. From that reason we assume that there exists a component α_i of the vector α which is responsible for the generation of an intrinsic RO of (7) at $\alpha = \alpha_0$. That means that the reduced system (7) can be rewritten in the form

$$\frac{dx_1}{d\sigma} = \bar{f}_1(x_1,x_2,\lambda), \qquad \lambda\frac{dx_2}{d\sigma} = \bar{f}_2(x_1,x_2,\lambda)$$

such that the functions $\bar{f}_1$, $\bar{f}_2$ guarantee the existence of a unique stable (unstable) RO near some closed curve Γ_0 for sufficiently small λ. The corresponding RO of the full system (1) is called a lifted RO. It is obvious that a system (1) with a lifted RO has at least three time-scales.

As an application of this approach we consider the existence of relaxation wave trains in (2) for small $\varepsilon := \nu/c^2$. This problem is equivalent to the existence of a RO of the system (3). The corresponding reduced system reads

$$\frac{dU}{d\tau} = f(U,\lambda) .$$

Let f be defined by

(8) $$f_1(u_1,u_2) := u_2 ,$$
$$f_2(u_1,u_2) := \lambda(1-u_1^2)u_2 - u_1 , \quad \lambda > 0 .$$

In this case - it represents the van der Pol oscillator with diffusion - the reduced system is equivalent to van der Pol's equation

(9) $$\frac{d^2y}{dt^2} + \lambda[y^2-1]\frac{dy}{dt} + y = 0 .$$

Setting $\tau = \lambda\vartheta$, $\varkappa = \lambda^{-2}$ (9) is equivalent to the system

$$\frac{dx}{d\vartheta} = -y , \qquad \varkappa\frac{dy}{d\vartheta} = x - \frac{y^3}{3} + y ,$$

whose right hand side satisfies the theorem of Mishchenko and Rozov [9], that is (9) has for small $\varkappa$ a unique stable intrinsic RO (see also [10]). Thus, the full system (3) with f defined in (8) has a unique stable lifted RO for $0 < \varepsilon < \varepsilon(\varkappa)$. The same approach yields a unique stable RO in the Oregonator model of the Belousov-Zhabotinskii -reaction [4,13] and in Nobel's model of the Purkinje fiber [2].

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