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Miloslav Feistauer; Alexander Ženíšek
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FINITE ELEMENT SOLUTION OF NONLINEAR ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS AND APPROXIMATIONS IN SOBOLEV-SLOBODECKIJ SPACES

FEISTAUER M., PRAGUE, Czechoslovakia
ŽENÍŠEK A., BRNO, Czechoslovakia

In this paper we present a complete theory of approximations of nonlinear elliptic problems with discontinuous coefficients by linear conforming triangular finite elements.

1. Continuous problem

Let us consider the following variational problem:
Find u: $\bar{\Omega} \to R^1$ such that

(1.1) a) $u - u^* \in V$, b) $a(u,v) = L(v) \quad \forall\, v \in V$.

Here V is a subspace of $H^1(\Omega) = W^{1,2}(\Omega)$, $u^* \in W^{1,p}(\Omega)$, $p > 2$, Ω is a bounded domain, a: $[H^1(\Omega)]^2 \to R^1$, $a(u,.) \in (H^1(\Omega))^*$ for each $u \in H^1(\Omega)$, $L \in V^*$. (1.1) represents a weak formulation of a boundary value problem for the equation

$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \Big(a_i(x,u(x),\nabla u(x))\Big) + a_0(x,u(x),\nabla u(x)) = f(x) \text{ in } \Omega$$

with discontinuous coefficients. This means that $\bar{\Omega}$ is decomposed into sets $\bar{\Omega}_1,\dots,\bar{\Omega}_m$ such that $\Omega_1,\dots,\Omega_m$ are mutually disjoint domains with Lipschitz-continuous piecewise C^3 boundaries $\partial\Omega_1,\dots,\partial\Omega_m$. For each $k = 1,\dots,m$ the coefficients $a_i = a_i^k$ in $\Omega_k \times R^3$ and $f = f^k$ in Ω_k. Across $\Gamma_{rs} = \partial\Omega_r \cap \partial\Omega_s$, $r \neq s$, a_i are discontinuous. For simplicity let us set $m = 2$. On $\partial\Omega$ we consider mixed Dirichlet-Neumann conditions

$$u|\Gamma_D = u_D, \quad \sum_{i=1}^{2} a_i(.,u,\nabla u)n_i = \varphi_N \text{ on } \Gamma_N,$$

where $\bar{\Gamma}_D \cup \bar{\Gamma}_N = \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$, $u_D = u^*|\Gamma_D$ and φ_N is piecewise C^2 on $\bar{\Gamma}_N$. On Γ_{rs} the so-called transition conditions are considered (see [4, 10]).

By assumptions on a_i we distinguish two cases: a) general *pseudo-monotone* case, when the form a(u,v) is Lipschitz-continuous, coercive and satisfies the generalized condition (S) (cf. [6]) and (1.1) has at least one solution; b) *strongly monotone* case with a Lipschitz-continuous and strongly monotone form a(u,v) and a unique solution to (1.1).

2. Discrete problem

Let us consider systems $\{\Omega_h\}$ and $\{\Omega_{sh}\}$ $(h \in (0,h_0)$, $s = 1,2$, $h_0 > 0)$ of polygonal approximations of Ω and Ω_s, respectively, $\bar{\Omega}_h = \bar{\Omega}_{1h} \cup \bar{\Omega}_{2h}$, $\Omega_{1h} \cap \Omega_{2h} = \emptyset$. Let $\mathcal{T}_h$ and $\mathcal{T}_{sh}$ denote triangulations of Ω_h and Ω_{sh}, respectively, with usual properties. By σ_h and σ_{sh} we denote the sets of all vertices of $\mathcal{T}_h$ and $\mathcal{T}_{sh}$, respectively. We assume that $\mathcal{T}_h = \mathcal{T}_{1h} \cup \mathcal{T}_{2h}$, $\sigma_h \subset \bar{\Omega}$, $\sigma_h \cap \partial\Omega_h \subset \partial\Omega$, $\sigma_{sh} \cap \partial\Omega_{sh} \subset \partial\Omega_s$, $\bar{\Gamma}_D \cap \bar{\Gamma}_N \subset \sigma_h$ and the points from $\partial\Omega_1 \cup \partial\Omega_2$ where either the condition of C^3-smoothness of $\partial\Omega_s$ or the condition of C^2-smoothness of φ_N are not satisfied are elements of σ_h. Let the system $\{\mathcal{T}_h\}$, $h \in (0,h_0)$ be *regular* ([1]).

Approximate solutions of (1.1) are sought in $X_h = \{v \in H^1(\Omega_h);\ v|T$ is affine $\forall\, T \in \mathcal{T}_h\}$. The space V is approximated by a suitable subspace $V_h \subset X_h$ and we set $u_h^* = r_h u^* =$ Lagrange interpolate of u^*. The forms a and L are approximated by

$$\tilde{a}_h(u,v) = \sum_{s=1}^{2} \int_{\Omega_{sh}} \left[\sum_{i=1}^{2} a_i^s(.,u,\nabla u)\frac{\partial v}{\partial x_i} + a_0(.,u,\nabla u)v\right]dx,$$

$$\tilde{L}_h(v) = \sum_{s=1}^{2} \int_{\Omega_{sh}} f^s v dx + \int_{\Gamma_{Nh}} \varphi_{Nh} v dS.$$

(We assume that a_i^s and f^s are defined on $\tilde{\Omega}_s \supset \bar{\Omega}_s$.) Further, the integrals in $\tilde{a}_h$ and $\tilde{L}_h$ are evaluated by numerical quadratures which yield the forms $a_h\colon X_h \times X_h \to R^1$ and $L_h\colon V_h \to R^1$ and we come to the *discrete problem* used in practice: Find $u_h\colon \bar{\Omega}_h \to R^1$ such that

(2.1) a) $u_h - u_h^* \in V_h$, b) $a_h(u_h,v_h) = L_h(v_h) \quad \forall\, v_h \in V_h$.

(For details see [4, 10]).

By the techniques from [5, 6] we get

2.2. Theorem. *For each* $h \in (0,h_0)$ *problem* (2.1) *has at least one solution* $u_h \in X_h$, *which is unique in the strongly monotone case. There exists a constant* $c > 0$ *such that*

(2.3) $$\|u_h\|_{1,\Omega_h} \le c \quad \forall\, h \in (0,h_0).$$

($\|\cdot\|_{1,\Omega_h}$ *denotes the norm in* $H^1(\Omega_h)$.)

3. Convergence

By [6] the approximate solutions $u_h \in X_h$ can be associated with their suitable modifications $\hat{u}_h \in H^1(\Omega)$ satisfying the estimate $\|\hat{u}_h\|_{1,\Omega} \le \hat{c}$ for all $h \in (0,h_0)$ (with $\hat{c}$ independent of h). Hence, we can choose sequences

(3.1) $$h_n \to 0+ \quad \text{and} \quad \hat{u}_{h_n} \to u \quad \text{weakly in } H^1(\Omega).$$

Let $u_c \in H^1(R^2)$ denote the Calderon extension of $u \in H^1(\Omega)$. The convergence results are contained in the following theorems:

3.2. Theorem (general pseudomonotone case). *If* (3.1) *holds, then* $\|u_{h_n}-u_c\|_{1,\Omega_{h_n}} \to 0$ *and* u *is a solution of* (1.1).

Proof - see [4].

3.3. Theorem (strongly monotone case without regularity). *If the form* a *is strongly monotone, then*

$$\lim_{h\to 0+} \|u_h-u_c\|_{1,\Omega_h} = 0.$$

Proof was carried out independently in [4] and [10].

Now let us consider the *strongly monotone case* provided the *exact solution is regular,* i. e.,

(3.4) $$u^s = u|\Omega_s \in H^2(\Omega_s), \quad s = 1,2.$$

Let $u^s_{cc} \in H^2(R^2)$ be a Calderon extension of u^s. Then we define an extension $\tilde{u}: \Omega \cup \Omega_h \to R^1$ of u:

(3.5) $$\tilde{u} = u \text{ on } \Omega, \quad \tilde{u} = u^s_{cc} \text{ on } \Omega_{sh} - \Omega, \quad s = 1,2,\ h \in (0,h_0).$$

3.6. Theorem. *If* (3.4) *is valid, then there exists a constant* $c > 0$ *such that*

$$\|\tilde{u}-u_h\|_{1,\Omega_h} \le c\,h, \quad h \in (0,h_0).$$

Proof was carried out independently in [4], with the "triple application of Green's theorem" (first used in [2]) and the separation of discretization and numerical integration errors, and in [10], on the basis of the approach from [9, 5] without the use of Green's theorem.

The approach from [10] has the importance in case of a *weak regularity* of the exact solution:

(3.7) $$u^s = u|\Omega_s \in H^{1+\varepsilon}(\Omega_s), \quad s = 1,2.$$

Here $H^{1+\varepsilon}(\Omega_s) = W^{1+\varepsilon,2}(\Omega_s)$, $\varepsilon \in (0,1)$, denotes a Sobolev-Slobodeckii space ([7, 8]).

3.8. Theorem. *Provided* (3.7) *holds and* $\tilde{u}$ *is defined by a relation analogous to* 3.5, *where* u^s_{cc} *is replaced by the appropriate extension* u^s_B *of* u^s *in* $H^{1+\varepsilon}(R^2)$, *there exists* $c > 0$ *such that*

$$\|u_h-\tilde{u}\|_{1,\Omega_h} \le c\,h^{\varepsilon}, \quad h \in (0,h_0).$$

Proof is based on the following interpolation result from [3]

$$\|v-r_h v\|_{1,\Omega_h} \le c\,h^{\varepsilon}\,\|v\|_{1+\varepsilon,\Omega_h}, \quad v \in H^{1+\varepsilon}(\Omega_h)$$

and the estimate from [10]

$$\|u^s_B\|_{1,\Omega_{sh}-\bar{\Omega}_s} \le c\,h^{\varepsilon}\,\|u^s\|_{1+\varepsilon,\Omega_s}$$

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