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FUNCTIONALS DEPENDING ON CURVATURES AND SURFACES WITH CURVATURE MEASURES

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The object of this talk is to discuss a possible approach by Direct Methods to the problem of the existence of minimizers of certain functionals $F(M)$, defined on surfaces M and depending on the curvatures of M. Functionals involving curvatures of curves and surfaces have been considered from the very begining of the Calculus of Variations in connection with the Theory of Elasticity and Differential Geometry, however we are not aware of any other approach by Direct Methods, in multidimensional problems, besides the recent works of Hutchinson and Leon Simon (see[Hu1],[Hu2] and [Si2]).

Here we shall consider only simple typical situations and we shall only state results without proof. For more general situations, precise statements and for the proofs, we refer to [AST].

In what follows M is a 2-dimensional smooth compact, oriented, embedded surface in $\mathbf{R}^3$, possibly with boundary, and $\nu(x)$ is a unit normal smooth vector field. We denote $H(x)$ and $K(x)$ the mean and the Gauss curvature respectively.

Here are two typical problems of interest.

Problem 1. *minimize the functional*

$$W(M) = \int_M |H(x)|^2 d\mathcal{H}^2(x)$$

among the closed surfaces M with a fixed genus g.
Problem 2. *Given a fixed surface M , minimize a functional of the type*

$$F(M) = \int_M f(H(x), K(x)) d\mathcal{H}^2(x)$$

among the surfaces $M \in \mathcal{M}$ such that

$$\begin{cases} \partial M = \partial M_0 \\ \nu_M(x) = \nu_{M_0}(x) \quad \text{on } \partial M_0 \end{cases} .$$

Problem 1 above is the well known Willmore's problem [Wil] [Pi.Ste], while problem 2 is a possible model for the equilibrium of an elastic shell [Tr].

The starting point of our approach is the simple remark below. We denote

$$G = \{(x, \nu(x)) | x \in M\} \subset M \times S^3 \subset \mathbf{R}^2_x \times \mathbf{R}^3_y$$

the graph of the Gauss map $\nu : M \to S^2$ and we denote $K_1(x), K_2(x)$ the principal curvatures of M at x, i.e. the eigenvalues of $d\nu(x)$.

Remark. One has

$$\mathcal{H}^2(G) = \int_M \{1 + K_1^2 + K_2^2 + (K_1 K_2)^2\}^{1/2}$$

hence the area of G is bounded by the sum of the area of M and the L^1-norms of the mean and the Gauss curvature.

Now consider for example problem 2 above and assume that the integrand f satisfies the coerciveness condition

$$f(H,K) \geq c_0(1+|H|+|K|) .$$

Then it is clear that $\mathcal{H}^2((\tau) \leq c_1\mathcal{F}(M)$ for all M and it follows that, if M_j is a minimizing sequence, one has $\mathcal{H}^2(G_j)+\mathcal{H}^2(\partial G_j) \leq$ const for all j. Hence [Fe.Fl.],[Fe] (possibly taking a subsequence) one has that the G_j converge weakly as currents to some integral current Σ in $\mathbf{R}^3 \times S^2$, i.e. $G_j(\phi) \rightarrow \Sigma(\phi)$ for all 2-forms ϕ with compact support in $\mathbf{R}^3 \times S^2$. Now this current Σ is a good candidate to be a generalized minimizer of our problem.

Obviously there are several questions:

(i) what are the properties of the integral currents Σ obtained as limits of smooth Gauss Graphs with uniformly bounded areas and L^1-norms (or L^p-norms) of the curvatures?

(ii) how does one extend the classical functionals to this large space of "generalized surfaces"?

(iii) what are the properties of the minimizers, under suitable assumptions for the functionals?

While the regularity question (iii) is completely open, there are some partial answers and conjectures [AST] for questions (i) and (ii). Here we shall consider only (i).

We remind to the reader that the integral current Σ is a functional on 2-forms of the type

$$\Sigma(\phi) = \int_R \langle \eta(x,y), \phi(x,y)\rangle \, \theta(x,y) d\mathcal{H}^2(x,y),$$

where R is a 2-dimensional $\mathcal{H}^2$-countably rectifiable set in $\mathbf{R}^3 \times S^2$, $\eta(x,y)$ is a unit tangent 2-vector field asociated to R and θ is an integer valued $\mathcal{H}^2$-measurable multiplicity function. Now, the nicest possible case is when one has $R = G$, $\theta = 1$ and the projection pR of R on $\mathbf{R}^3$ is M. This does not happen in general, as very simple examples show, however it can be proved [AST] that the rectifiable set R is always the union of a "vertical part" and of the graph over the rectifiable set pR of a unit normal (possibly double valued) vector field $(\pm)\nu$. Moreover, the normal field ν is approximately differentiable on pR. Finally, for the "generalized surfaces" Σ one can define suitable generalized notions of second fundamental form and curvatures. This generalized curvatures are measures in $\mathbf{R}^3 \times S^2$. If Σ is closed and compactly supported one can define the Euler-Poincarè characteristic of Σ as the value on the whole space of the generalized Gauss curvature.

We end with a few remarks. Our description of the curvatures via the graph of the Gauss map is in many respect similar to the description by M.Zähle [Zä] via the unit normal bundle. However we find our formulation more convenient, partly because it makes very simple the statement of boundary value problems and partly because it is a natural outgrowth of well known techniques as varifolds ([Alm],[Si1]) and Cartesian currents ([GMS1],[GMS2]). In fact the idea of thinking of a smooth surface M as a suitable measure supported on the graph G of the Gauss map of M is a classical one (we refer to Young [You1] and Almgren [Alm] where, actually, the unoriented case is considered). While these authors considered a *positive* measure on G, corresponding to the area of M, here we consider the *vector valued* measure $\eta\mathcal{H}^2 \llcorner G$ (where η is the 2-tangent vector to G), that is the current carried by G. Of course, in this approach we have been influenced by the recent works [GMS1] and [GMS2]. We should like to stress on the fact that, as in [GMS1] and [GMS2], one main characteristic of our approach is that one can work with integral functional of the curvatures such that the integrand is a convex function of the exterior powers (i.e. minors) of all order of the second fundamental form, hence a non necessarily-convex function of the second fundamental form.

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