FINITE ELEMENT METHODS FOR LINEAR COUPLED THERMOELASTICITY

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According to [1], the statical two-dimensional problem of linear coupled thermoelasticity can be formulated in the following way: Let Ω be a bounded domain in the x_1, x_2-plane with a sufficiently smooth boundary Γ. Find a displacement vector $\underline{u}(x_1, x_2, t)$ and a temperature $T(x_1, x_2, t)$ which satisfy the following equations and boundary and initial conditions:

(1) $T_{,ii} + Q = c_1 \dot{T} + c_2 T_r \dot{u}_{j,j}$ in $\Omega \times (0, t^*]$

(2) $\sigma_{ij,j} + X_i = 0 \quad (i = 1,2)$ in $\Omega \times (0, t^*]$

(3) $T(x_1, x_2, t)|_{\Gamma} = \bar{T}(x_1, x_2), \quad t > 0$

(4) $u_i(x_1, x_2, t)|_{\Gamma_1} = \bar{u}_i(x_1, x_2) \quad (i = 1,2), \quad t > 0$

(5) $\sigma_{ij} \nu_j |_{\Gamma_2} = p_i(x_1, x_2) \quad (i = 1,2), \quad t > 0$

(6) $T(x_1, x_2, 0) = T_0(x_1, x_2), \quad (x_1, x_2) \in \Omega$

(7) $u_i(x_1, x_2, 0) = u_{i0}(x_1, x_2) \quad (i = 1,2), \quad (x_1, x_2) \in \Omega$

where

(8) $\sigma_{ij} = D_{ijkl} [\varepsilon_{kl} - \alpha (T - T_r) \delta_{kl}]$

(9) $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$

(10) $D_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \geq \mu_0 \varepsilon_{ij} \varepsilon_{ij} \quad \forall \varepsilon_{ij}, \ \mu_0 = \text{const} > 0.$

A summation convention over a repeated subscript is adopted. A comma is employed to denote partial differentiation with respect to spatial coordinates and a dot denotes the derivative with respect to time. Thus equation (1) is the coupled heat equation; the symbol Q denotes a prescribed rate of internal heat generation per unit volume, c_1 and c_2 are positive constants depending only on the material of a considered body and T_r is a positive constant which has the meaning of the temperature for which the material is stress-free. Equations (2) are Cauchy's equations of equilibrium, the symbols X_1, X_2 denote prescribed components of body forces per unit volume. The functions on the right-hand sides of relations (3) - (7) are prescribed functions. The symbols Γ_1, Γ_2 denote two disjoint subsets of Γ such that mes $\Gamma_1 > 0$ and $\Gamma = \Gamma_1 + \Gamma_2$. In equation (5), ν_1 and ν_2 are components of the outward unit normal to Γ. In equation (8), α is the coefficient of linear thermal expansion, δ_{ij} is the Kronecker delta and D_{ijkl} are material constants. We consider isotropic materials only.

In what follows we shall suppose that the problem (1) - (10) has a solution $\underline{u}$, T. Then, according to [1, pp. 38 - 40], this solution is unique.

We shall solve the problem (1) - (10) by the finite element method using curved triangular elements and numerical integration. We approximate the domain Ω by a domain Ω_h the boundary Γ_h of which consists of arcs of degree n. These arcs are the curved sides of curved triangles. On the triangulation $\mathcal{T}_h$ of Ω_h we shall construct two finite element spaces V_h and W_h which are finite dimensional subspaces of $C^0(\Omega_h)$. For a given $t = t_m$ the displacement field $\underline{u}$ will be approximated in the space $V_h \times V_h$ and the temperature field T in the space W_h.

In applications we usually choose n = 3. In this case the boundary Γ can be approximated piecewise either by arcs of the Hermite type or by arcs of the Lagrange type. The construction of the corresponding spaces V_h can be found in [2], [4], [5]. The spaces V_h have the following interpolation property: If $f \in H^{n+1}(\Omega_h)$ and $f_I \in V_h$ is the interpolate of f then

$$\| f - f_I \|_{j,\Omega_h} \leq Ch^{n+1-j} \| f \|_{n+1,\Omega_h} \quad (j = 0,1)$$

where the constant C does not depend on h and f.

In the case of curved elements the construction of the space W_h depends on the choice of the space V_h. We choose $p < n$ (usually $p = n - 1$) and construct the space W_h in such a way (details are described in [8]) that it has the following interpolation property: If $f \in H^{p+1}(\Omega_h)$ and $f_I \in W_h$ is the interpolate of f then

$$\| f - f_I \|_{j,\Omega_h} \leq Ch^{p+1-j} \| f \|_{p+1,\Omega_h} \quad (j = 0,1).$$

It should be noted that in the case of polygonal boundary Γ the spaces V_h and W_h can be constructed quite indepedently.

It is well-known that all numerical computations in the case of both curved and classical triangles are carried out on the unit triangle K_0 which lies in the ξ_1,ξ_2-plane and has the vertices (0,0), (1,0), (0,1) (see, e.g., [2], [5], [6]). Let us choose on K_0 and on the segment [0,1] certain quadrature formulas (see Theorem 1) and using them let us compute approximately the integrals

$$\tilde{D}_h(v,w) = \int_{\Omega_h} v_{,i} w_{,i} \, dx \,, \quad (v,w)_{0,\Omega_h} = \int_{\Omega_h} vw \, dx \,,$$

$$\tilde{a}_h(\underline{v},\underline{w}) = \int_{\Omega_h} D_{ijkl} \, \varepsilon_{ij}(\underline{v}) \varepsilon_{kl}(\underline{w}) \, dx \,,$$

$$(\underline{v},\underline{w})_{0,\Omega_h} = \int_{\Omega_h} v_i w_i \, dx \, , \quad \langle \underline{p}_h, \underline{v} \rangle_{\Gamma_{h2}} = \int_{\Gamma_{h2}} p_{hi} v_i \, ds$$

where p_{hi} denotes the function which we obtain by "transferring" the function p_i from the curve Γ_2 onto the curve Γ_{h2} (details are in [6]), Γ_{h2} being the approximation of Γ_2. Then we obtain the forms $D_h(v,w)$, $(v,w)_h$, $a_h(\underline{v},\underline{w})$, $(\underline{v},\underline{w})_h$, $\langle \underline{p}_h, \underline{v} \rangle_h$.

Further, let us define the sets

$$V_{h0} = \left\{ v \in V_h : \ v = 0 \text{ on } \Gamma_{h1} \right\}, \quad V_{hu}^i = \left\{ v \in V_h : \ v = \bar{u}_i^{apr} \text{ on } \Gamma_{h1} \right\},$$

$$W_{h0} = \left\{ w \in W_h : \ w = 0 \text{ on } \Gamma_h \right\}, \quad W_{hT} = \left\{ w \in W_h : \ w = \bar{T}^{apr} \text{ on } \Gamma_h \right\}$$

where Γ_{h1} is the approximation of Γ_1 and $\bar{u}_i^{apr} \in V_h$ and $\bar{T}^{apr} \in W_h$ are the interpolates of the functions $\bar{u}_i$ and $\bar{T}$, respectively.

Let us choose an integer M, set $\Delta t = t^*/M$ and define $t_m = m\Delta t$ $(m = 0,1,\dots,M)$. Let us use the notation $f^m \equiv f^m(x_1,x_2) = f(x_1,x_2,m\Delta t)$. If we use one-step A-stable methods for the time discretization then we can define the discrete problem for approximate solving the variational problem which corresponds to the problem (1) - (10) in the following way:

For each $m = 0,1,\dots,M-1$ find a vector $\underline{u}_h^{m+1} \in V_{hu}^1 \times V_{hu}^2$ and a function $T_h^{m+1} \in W_{hT}$ such that

(11) $$\Delta t D_h\Big(\sum_{j=0}^{1} \beta_j T_h^{m+j}, w\Big) + c_1\Big(\sum_{j=0}^{1} \alpha_j T_h^{m+j}, w\Big)_h + c_2 T_r\Big(\sum_{j=0}^{1} \alpha_j u_{hi,i}^{m+j}, w\Big)_h = \Delta t\Big(\sum_{j=0}^{1} \beta_j Q^{m+j}, w\Big)_h \quad \forall w \in W_{h0}$$

(12) $$a_h\Big(\sum_{j=0}^{1} \beta_j \underline{u}_h^{m+j}, \underline{v}\Big) - c_3\Big(\sum_{j=0}^{1} \beta_j T_h^{m+j} - T_r, v_{i,i}\Big)_h = \Big(\sum_{j=0}^{1} \beta_j \underline{X}^{m+j}, \underline{v}\Big)_h + \langle \underline{p}_h, \underline{v} \rangle_h \quad \forall \underline{v} \in V_{h0} \times V_{h0}$$

(13) $$\underline{u}_h^0 = \underline{u}_0^{apr}(x_1,x_2) \, , \quad T_h^0 = T_0^{apr}(x_1,x_2)$$

where c_3 is a constant depending only on D_{ijkl} and α, $T_0^{apr} \in W_h$ is an approximation of the function T_0, $\underline{u}_0^{apr} \in V_h \times V_h$ is an approximation of the vector $\underline{u}_0$ and

(14) $$\alpha_0 = -1, \ \alpha_1 = 1, \ \beta_0 = \Theta, \ \beta_1 = 1 - \Theta$$

where $\Theta \le 1/2$ is any real number.

<u>Theorem 1</u>. Let the boundary Γ be of class C^{n+1}. Let every triangulation $\mathcal{T}_h$ satisfy the condition $\bar{h}/h \geq c_0$, where $c_0 = \text{const} > 0$, $\bar{h} = \min_{K \in \mathcal{T}_h} h_K$ and $h = \max_{K \in \mathcal{T}_h} h_K$. Let a quadrature formula on the unit

triangle K_0 for calculation of the form $D_h(v,w)$ be of degree of precision $2p - 1$. Let quadrature formulas on K_0 for calculation of the forms $(v,w)_h$, $(\underline{v},\underline{w})_h$ and $a_h(\underline{v},\underline{w})$ be of degree of precision $2n - 2$. Let a quadrature formula on the unit segment $[0,1]$ for calculation of the form $\langle \underline{p}_h, \underline{v} \rangle_h$ be of degree of precision $2n - 1$. Let the exact solution T, $\underline{u}$ of the problem (1) - (10) satisfy $\partial^k T/\partial t^k \in L^\infty(H^{p+3}(\Omega))$, $\partial^k u_i/\partial t^k \in L^\infty(H^{n+1}(\Omega))$ $(k = 0,\dots,q+1,$ $i = 1,2)$ where q is the order of the Θ-method ($q = 1$ for $\Theta < 1/2$, $q = 2$ for $\Theta = 1/2$). Let $Q \in L^\infty(H^{p+1}(\Omega))$, $X_i \in L^\infty(H^n(\Omega))$. Then for sufficiently small h there exists one and only one solution T_h^m, $\underline{u}_h^m$ $(m = 1,\dots,M)$ of the problem (11) - (14) and it holds

$$\|\tilde{\underline{u}}^m - \underline{u}_h^m\|_{1,\Omega_h} + \|\tilde{T}^m - T_h^m\|_{0,\Omega_h} \leq C\left(\Delta t^q + h^{p+1} + h^n + s_0\right)$$

where C is a constant independent on h and Δt, $\tilde{\underline{u}}$ and $\tilde{T}$ are the Calderon extensions of $\underline{u}$ and T, respectively, and

$$s_0 = \|\underline{u}_h^0 - \underline{r}^0\|_{1,\Omega_h} + \|T_h^0 - \eta^0\|_{0,\Omega_h}$$

$\underline{r}$ and η being the Ritz approximations of $\tilde{\underline{u}}$ and $\tilde{T}$, respectively.

Theorem 1 is proved in [8]. The proof is a generalization of devices used in [3],[6] and [7]. The obtained result can be extended to the case of two-step A-stable methods.

References

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