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GALERKIN-FINITE ELEMENT SOLUTION OF NONLINEAR EVOLUTION PROBLEMS

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1. Introduction

The generalized Galerkin method for the solution of evolution problems consists of the following steps: 1) We formulate the given problem in a variational form. 2) We discretize the problem in space, i.e. we consider a family $\{V^h\}$, $0 < h < h^*$, of finite dimensional subspaces of the basic Banach space V such that $\lim_{h \to 0_+} \text{dist}\,(V^h, v) = 0$ $\forall\, v \in V$ (see Nečas [10] p. 47) and in V^h we define a semidiscrete solution by means of a discrete analog of the variational formulation determining the exact solution. 3) To compute this solution means to solve a system of ordinary differential equations. Solving this system numerically we get a completely discretized approximate solution. In case of nonlinear problems the application of linear multistep methods has advantage in that we are often able to linearize the resulting scheme without lowering the accuracy. We restrict ourselves to a narrow class of linear multistep methods: to A-stable methods. These methods lead to unconditionally stable schemes fulfilling certain energy inequalities. Both these properties are desirable, the other providing a simple way for the derivation of apriori error estimates.

First we describe a class of linear multistep methods considered in the sequel. If we apply linear multistep methods to the scalar equation $\dot{x} = f(x, t)$ we get a scheme of the form

$$(1.1)\quad \sum_{j=0}^{k} \alpha_j x^{m+j} = \Delta t \sum_{j=0}^{k} \beta_j f^{n+j}, \quad f^n = f(x^n, t_n), \quad t_n = n\Delta t.$$

According to a classical result of Dahlquist [3] A-stable methods cannot have a greater order of accuracy than 2. Therefore, we restrict ourselves to one- and two-step methods (k = 1,2). Normalized by $\sum_{j=0}^{k} \beta_j = 1$ all such A-stable methods are (see Liniger [8])

$$\alpha_1 = 1, \alpha_0 = -1, \beta_1 = 1 - \Theta, \beta_0 = \Theta, \Theta \leq \tfrac{1}{2} \text{ if } k = 1,$$

$$(1.2)\quad \alpha_1 = 1 - 2\alpha_2, \alpha_0 = -1 + \alpha_2, \beta_1 = \tfrac{1}{2} + \alpha_2 - 2\beta_2, \beta_0 = \tfrac{1}{2} - \alpha_2 + \beta_2,$$

$$\alpha_0 \geq \tfrac{1}{2}, \beta_2 > \tfrac{1}{2}\alpha_2 \quad \text{if } k = 2.$$

The order q is equal to 1 if k = 1 and $\vartheta < \frac{1}{2}$ and 2 if k = 1 and $\Theta = \frac{1}{2}$ or k = 2. Three best known special schemes are

$$(1.3)\quad \begin{aligned} &x^{n+1} - x^n = \Delta t\, f^{n+1},\ x^{n+1} - x^n = \tfrac{1}{2}\Delta t\left[f^{n+1} + f^n\right],\\ &\tfrac{3}{2} x^{n+2} - 2\, x^{n+1} + \tfrac{1}{2} x^n = \Delta t\, f^{n+2}. \end{aligned}$$

The schemes (1.2) fulfill energy inequalities. Let V be a vector space and b(u, v) be a bilinear symmetric form on $V \times V$. We assume that b(u, v) is nonnegative, i.e.

$$0 \leq b(u,\ u) \equiv |u|^2 \quad \forall u \in V.$$

Consider the sequence

$$S^n = b\Big(\sum_{j=0}^{k} \alpha_j\, u^{n+j},\ \sum_{j=0}^{k} \beta_j\, u^{n+j}\Big)$$

where $u^n \in V$, n = 0,1,... . Then (see Zlámal [12] and [13])

$$(1.4)\quad |u^m|^2 \leq C_1 \sum_{j=0}^{k-1} |u^j|^2 + C_2 \sum_{n=0}^{m-k} S^n, \quad m \geq k.$$

Here C_1, C_2 are positive constants depending on the coefficients α_j, β_j only. The two important choices of V are: 1) $V = L^2(\Omega)$ and $b(u, v) = (u, v)_{L^2(\Omega)}$; 2) V is a subspace of $H^1(\Omega)$ and $b(u, v) = \int_\Omega \sum_{i,j=1} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx$, $\left\{a_{ij}(x)\right\}_{i,j=1}^{N}$ being a positive definite matrix.

2. A nonlinear heat equation

As a first example we consider the nonlinear heat equation

$$(2.1)\quad \frac{\partial u}{\partial t} = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left[k_{ij}(x,\ t,\ u)\,\frac{\partial u}{\partial x_j}\right] + f(x,\ t,\ u) \quad \text{in } \Omega\times(0,\ T),\quad 0 < T < \infty,$$

with boundary and initial conditions

$$u(x,\ t) = 0 \ \text{ on } \partial\Omega\times(0,\ T),\quad u(x,\ 0) = u^0(x) \ \text{ in } \Omega.$$

Here $\Omega \subset R^N$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $x = (x_1, \dots, x_N)$, $\left\{k_{ij}(x, t, u)\right\}_{i,j=1}^{N}$ is a uniformly positive definite matrix and $k_{ij}(x, t, u)$ and $f(x, t, u)$ are supposed to be uniformly Lipschitz continuous functions of $t \in [0, T]$ and of $u \in (-\infty, \infty)$.

If the exact solution is smooth enough then it holds

$$(2.2)\quad (u',\ v)_0 + a(t;\ u;\ u,\ v) = (f(x,\ t,\ u),\ v)_0 \ \text{ in } [0,\ T]\ \forall v \in H_0^1(\Omega).$$

Here

$$u' = \frac{\partial u}{\partial t}, \quad (u, v)_0 = \int_\Omega u\, v\, dx,$$

$$a(t, w; u, v) = \int_\Omega \sum_{i,j=1}^{N} k_{ij}(x, t, w) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx,$$

$H^m(\Omega)$ is the Sobolev space $\{u \in L^2(\Omega);\ D^\alpha u \in L^2(\Omega)\ \ \forall |\alpha| \leq m\}$ with the usual scalar product $(u, v)_m = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}$ and the norm $\|u\|_m = (u, u)_m^{\frac{1}{2}}$ and $H_0^1(\Omega) = \{u | u \in H^1(\Omega);\ u|_{\partial\Omega} = 0\}$.

Let us consider a family of finite element spaces such that $V^h \subset H_0^1(\Omega)$. The Galerkin method yields a semidiscrete solution $U(x, t)$ which for each $t \in [0, T]$ is a function from V^h. $U(x, t)$ is uniquely determined by a discrete analog of (2.2):

$$(2.3)\quad (U', v)_0 + a(t, U; U, v) = (f(x, t, U), v)_0 \quad \forall v \in V^h, \qquad U(x, 0) = U^0(x);$$

$U^0(x)$ is a suitable approximation of $u^0(x)$ from V^h. (2.3) represents a system of ordinary differential equations. Applying the method (1.2) we obtain

$$(2.4)\quad \begin{aligned} &\Big(\sum_{j=0}^{k} \alpha_j U^{n+j}, v\Big)_0 + \Delta t \sum_{j=0}^{k} \beta_j\, a(t_{n+j}, U^{n+j}; U^{n+j}, v) = \\ &= \Delta t \sum_{j=0}^{k} \beta_j (f(x, t_{n+j}, U^{n+j}), v)_0 \quad \forall v \in V^h,\ 0 \leq n \leq \frac{T}{\Delta t} - k. \end{aligned}$$

The scheme (2.4) being nonlinear has little practical value. We linearize it as follows:

$$(2.5)\quad \begin{aligned} &\Big(\sum_{j=0}^{k} \alpha_j U^{n+j}, v\Big)_0 + \Delta t \sum_{j=0}^{k} \beta_j\, a(t_{\bar{n}}, U^{\bar{n}}; U^{n+j}, v) = \\ &= \Delta t (f(x, t_{\bar{n}}, U^{\bar{n}}), v)_0 \quad \forall v \in V^h, \quad 0 \leq n \leq \frac{T}{\Delta t} - k; \end{aligned}$$

for k = 1

$$(2.6)\quad t^{\bar{n}} = \begin{cases} t_n, & \Theta < \frac{1}{2}, \\ t_n + \frac{1}{2}\Delta t, & \Theta = \frac{1}{2}, \end{cases} \qquad U^{\bar{n}} = \begin{cases} U^n, & \Theta < \frac{1}{2}, \\ \frac{3}{2} U^n - \frac{1}{2} U^{n-1}, & \Theta = \frac{1}{2}, \end{cases}$$

(see Douglas and Dupont [4]),
for k = 2

$$(2.7)\quad t_{\bar{n}} = t_n + (\tfrac{1}{2} + \alpha_2)\Delta t, \quad U^{\bar{n}} = (\tfrac{1}{2} + \alpha_2) U^{n+1} + (\tfrac{1}{2} - \alpha_2) U^n$$

(see Zlámal [12]). The order of accuracy q of the method (2.5) is equal 1 if k = 1 and $\vartheta < \frac{1}{2}$ and 2 if k = 1, $\vartheta = \frac{1}{2}$ or k = 2. Notice that whereas (2.4) with k = 1 and $\vartheta = \frac{1}{2}$ is a one-step scheme the corresponding scheme (2.5) is a two-step scheme.

Remark. Even when (2.5) represents a linear algebraic system at every time step it is not the final scheme in practical computations. In general, we have to consider finite element spaces V^h which are subspaces of $H_o^1(\Omega_h)$, $\Omega_h \neq \Omega$ (best known example: curved isoparametric elements). In addition, we have to compute mass and stiffness matrices numerically.

We assume that the family $\{V^h\}$ has the following approximation property shared by finite element subspaces: to any $u \in H^{p+1}(\Omega) \cap H_o^1(\Omega)$ there exists $u^h \in V^h$ such that

$$(2.8) \quad \|u - u^h\|_o + h\|u - u^h\|_1 \leq C\, h^{p+1}\|u\|_{p+1}.$$

Then if the exact solution is sufficiently smooth the following error bounds are true:

$$(2.9) \quad \|u^m - U^m\|_o \leq C\Big[\sum_{j=0}^{k-1} \|u^j - U^j\|_o + h^{p+1} + \Delta t^q\Big], \quad k \leq m \leq \frac{T}{\Delta t};$$

(q is the order of (1.2)). For one-step methods (2.9) was proved by Wheeler [11], for two-step methods by Zlámal [12] by means of the energy inequality (1.4).

The scheme (2.5) is linear, however one has to recompute the matrix arizing from the form a(w; u, v) at every time step. In recent years they have been proposed linear schemes which can be more effective. We refer to papers by Douglas, Dupont and Ewing [6] and by Bramble [1]. A different approach has been suggested by Crouzeix [2]. In fact, two special schemes of this kind were proposed much earlier by Douglas and Dupont [5].

The scheme (2.5) has been applied to the solution of the time dependent Navier-Stokes equations by Girauld and Raviart [7]. The error bounds have been derived again by means of the energy inequality (1.4).

3. Nonlinear quasistationary magnetic field

In recent years attention has been paid in electrical engineering journals to the computation of nonlinear quasistationary magnetic field. This problem occurs, e.g., in designing the magnet systems for fusion reactors and in rotating machinery. In two dimensions it can be formulated in the following model way: There is given a two-dimensional bounded domain Ω and an open nonempty set $R \subset \Omega$. We are looking for a function $u = u(x_1, x_2, t)$ (magnetic vector potential) such that

1)

(3.1) $\sigma \frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\left(\nu \frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\nu \frac{\partial u}{\partial x_2}\right) + J$ in $R \times (0, T)$,

(3.2) $u(x_1, x_2, 0) = u_0(x_1, x_2)$ in R,

2)

(3.3) $0 = \frac{\partial}{\partial x_1}\left(\nu \frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\nu \frac{\partial u}{\partial x_2}\right) + J$ in $S \times (0, T)$, $S = \Omega - \bar{R}$,

3) u satisfies a boundary condition on $\partial\Omega$,

4) u satisfies the conditions

(3.4) $[u]_R^S = \left[\nu \frac{\partial u}{\partial n}\right]_R^S = 0$ on $\Gamma = \partial R \cap \partial S$.

Here the conductivity $\sigma = \sigma(x_1, x_2)$ is a positive function on R, the reluctivity $\nu = \nu(x_1, x_2, \| \mathrm{grad}\, u\|)$, $\|\mathrm{grad}\, u\|^2 = \left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2$, is a positive function on $\Omega \times [0, \infty)$, $J = J(x_1, x_2, t)$ is a given current density, $u_0(x_1, x_2)$ is a given function defined on R and n is the normal oriented in a unique way.

The problems 1) - 4) can be easily formulated in a variational form. Let us, for simplicity, consider the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega.$$

We multiply (3.1) and (3.3) by a function $v \in H_0^1(\Omega)$, we integrate, we use Green's formula and (3.3) and we sum. The result is

(3.5) $\left(\sigma \frac{\partial u}{\partial t}, v\right)_{L^2(R)} + a(u, v) = (J, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega)$

where

$$a(u, v) = \int_\Omega \sum_{i=1}^{2} \nu \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx.$$

(3.5) is taken in Melkes, Zlámal [9] as the starting point for the construction of the approximate solution. Here, we briefly mention some results of Zlámal [14].

There are given two abstract equivalent formulations of the problem. One of them is a generalization of (3.5). Without all details, we are looking for a function $u \in W_R = \{u | u \in L^p(0, T; V);$ $u_R' \in L^{p'}(0, T; \bar{V}_R')\}$ statisfying

$$(3.6)\quad \frac{d}{dt}(u_R, v_R) + a(u, v) = \langle f, v\rangle \text{ in } \mathcal{D}'((0, T)) \quad \forall v \in V,$$
$$u(0)_R = u^o.$$

Here V is a separable reflexive Banach space, u_R is, roughly speaking, the restriction of u to R, $V_R = \{\omega | \omega = v_R, v \in V\}$, $a(u, v)$ is a hemicontinuous monotone form on $V \times V$ (linear in v and, in general, nonlinear in u) such that

$$a(u, u) \geq \alpha \|u\|_V^p, \quad |a(v, v)| \leq C\|u\|_V^{p-1}\|v\|_V \quad \forall u, v \in V,$$

$(\omega, z)_R$ is a scalar product of a Hilbert space H_R in which V_R is dense and continuously imbedded, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\langle f, v\rangle$ is the value of a functional from $L^{p'}(0, T; V')$. The fully discrete approximate solution $U^n \in V^h$ $(n = 1,2,\ldots)$ is defined in two ways:

$$(U_R^{n+1} - U_R^n, v_R)_R + \Delta t\, a(U^{n+1}, v) =$$
$$= \Delta t \langle f^{n+1}, v\rangle \quad \forall v \in V^h, \quad U_R^o = u^o,$$

or

$$(\tfrac{3}{2} U_R^{n+2} - 2 U_R^{n+1} + \tfrac{1}{2} U_R^n, v_R)_R + \Delta t\, a(U^{n+2}, v) =$$
$$= \Delta t \langle f^{n+2}, v\rangle \quad \forall v \in V^h, \quad U_R^{-1} = U_R^o = u^o$$

with $f^n = \Delta t^{-1} \int_{t_{n-1}}^{t_n} f(\tau)d\tau$. Evidently, there are used the first and the third scheme of (1.3) for time discretization. In both cases, the solutions (which uniquely exist) are extended to $(0, T]$ as step functions:

$$U^\delta = U^n \text{ in } (t_{n-1}, t_n], \quad n = 1,2,\ldots, \quad \delta = (h, \Delta t).$$

It is proved that the problem (3.6) has a unique solution and that

$$U^\delta \longrightarrow u \text{ in } L^p(0, T; V) \text{ weakly if } \delta \longrightarrow 0.$$

In the special case (3.5) all assumptions are fulfilled $(p = 2)$ if $\nu(x_1, x_2, \xi)$ possesses Caratheodory's property, is bounded from above and

$$\xi\nu(x_1, x_2, \xi) - \eta\nu(x_1, x_2, \eta) \geq \alpha(\xi - \eta) \quad \forall \xi \geq \eta \geq 0,$$
$$\alpha = \text{const} > 0.$$

It is also proved that if the exact solution is enough smooth in R and in S and the elements of V^h are piecewise linear functions then

$$\|u - U^\delta\|_{L^2(0,T;H^1(\Omega))} \leq C[h + \Delta t^q], \quad q = 1,2.$$

4. A damped nonlinear wave equation

Let $\{a_{ij}(x)\}_{i,j=1}^{N}$ be a uniformly positive definite matrix. Let $d(x, t, u, z)$ and $g(x, t, u, z)$ be piecewise continuous with respect to x and uniformly Lipschitz continuous with respect to t, u and z for $(x, t) \in \bar{\Omega} \times [0, T]$ and $u, z \in (-\infty, \infty)$. Further, we assume

$$d(x, t, u, z) \geq 0.$$

We consider the equation

$$(4.1) \quad \frac{\partial^2 u}{\partial t^2} + d(x, t, u, u') \frac{\partial u}{\partial t} = Lu + g(x, t, u, u') \text{ in } \Omega$$

where

$$Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} [a_{ij}(x) \frac{\partial u}{\partial x_j}], \quad u' = \frac{\partial u}{\partial t},$$

with the boundary and initial conditions

$$(4.2) \quad u = 0 \text{ on } \partial\Omega \times (0, T), \ u(x, 0) = u^o(x), \ u'(x, 0) = z^o(x) \text{ in } \Omega,$$
$$u^o, z^o \in H_0^1(\Omega).$$

We write the problems (4.1), (4.2) in a variational form; we set

$$(4.3) \quad u' = z$$

so that $z' = -d(x, t, u, z)z + Lu + g(x, t, u, z)$. If the exact solution is smooth enough then it follows

$$(4.4) \quad (z', v)_o = -(d(x, t, u, z)z, v)_o - a(u, v) + (g(x, t, u, z), v)_o \quad \forall v \in H_0^1(\Omega)$$

where

$$a(u, v) = \int_\Omega \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx.$$

The equations (4.3), (4.4) serve as the starting point for the construction of a fully descrete approximate solution.

First, we define a semidiscrete solution. As before let $\{V^h\}$, $0 < h \leq h^*$, be a family of finite dimensional subspaces of $H_0^1(\Omega)$ possessing the approximation property (2.8). By a semidiscrete Galerkin solution we mean a couple of functions $U(x, t), Z(x, t) \in V^h$ $\forall t \in [0, T]$ satisfying in $(0, T)$

$$U' = Z, \ (Z', v)_o = -(d(x, t, U, Z)Z, v)_o - a(U, v) + (g(x, t, U, Z), v)_o \ \forall v \in V^h, \ U(x, 0) = U^o(x), \ Z(x, 0) = Z^o(x).$$

Here U^o, $Z^o \in V^h$ are suitable approximations of u^o, z^o. The discretization in time is carried out by the scheme (1.2) and by the linearization procedure introduced in section 2 as follows:

$$\sum_{j=0}^{k} \alpha_j U^{n+j} = \Delta t \sum_{j=0}^{k} \beta_j Z^{n+j}, \ (\sum_{j=0}^{k} \alpha_j Z^{n+j}, v)_o =$$

$$= - \Delta t(d^{\bar{n}} \sum_{j=0}^{k} \beta_j Z^{n+j}, v)_o$$

$$- \Delta t \ a(\sum_{j=0}^{k} \beta_j U^{n+j}, v) + \Delta t(g^{\bar{n}}, v)_o \quad \forall v \in V^h;$$

here

$$d^{\bar{n}} = d(x, t_{\bar{n}}, U^{\bar{n}}, Z^{\bar{n}}), \ g^{\bar{n}} = g(x, t_{\bar{n}}, U^{\bar{n}}, Z^{\bar{n}})$$

and $t_{\bar{n}}$, $U^{\bar{n}}$, and in the same way $Z^{\bar{n}}$, are given by (2.6), (2.7).

It is easy to prove that to find U^{n+k}, Z^{n+k} means to solve a system of linear equations with a positive definite matrix. Hence U^n, $Z^n (k \leq n \leq \frac{T}{\Delta t})$ are uniquely defined.

The energy inequality (1.4) (used twice with $b(u, v) = (u, v)_o$ as well as with $b(u, v) = a(u, v)$) can be again successfully applied for deriving error estimates. We state the result for the case of ϑ-method with $\vartheta < \frac{1}{2}$ which is of order one ($q = 1$). Besides the hypotheses introduced above and besides some regularity conditions which we do not introduce we assume that U^o is the Ritz projection of u^o, i.e. $a(U^o, v) = a(u^o, v) \quad \forall v \in V^h$, and that $\|Z^o - z^o\|_o \leq C h^{p+1}$ (e.g., we can take the interpolate of z^o in V^h for Z^o). Then (see Zlámal [13])

$$\begin{aligned} \| u^m - U^m\|_o &\leq C(h^{p+1} + \Delta t), \\ \|u'^m - Z^m\|_o &\leq C(h^{p+1} + \Delta t), \qquad 1 \leq m \leq \Delta t^{-1} T, \\ \| u^m - U^m\|_1 &\leq C(h^p + \Delta t). \end{aligned}$$

References

[1] Bramble, J. H.: Multistep methods for quasilinear parabolic equations. J. T. Oden, Ed. Computational methods in nonlinear mechanics, North-Holland, 1980, 177 - 183.

[2] Crouzeix, M.: Une méthode multipas implicite-explicite pour l'approximation des equations paraboliques. Numer. Math. 35 (1980), 257 - 276.

[3] Dahlquist, G. A.: A special stability problem for linear multistep methods. BIT 3 (1963), 27 - 43.

[4] Douglas, J. jr. and T. Dupont: Galerkin methods for parabolic equations. SIAM J. Numer. Anal. 7 (1970), 575 - 626.

[5] Douglas, J. jr. and T. Dupont: Alternating-direction methods in rectangles. B. Hubbard, Ed. Numerical solution of partial differential equation II. Academic Press 1971.

[6] Douglas, J. jr., T. Dupont and R. E. Ewing: Incomplete iteration for time-stepping a Galerkin method for a quasilinear parabolic problem. SIAM J. Numer. Anal. 16 (1979), 503 - 522.

[7] Girault, V., P.-A. Raviart: Finite element approximation of the Navier-Stokes equations. Springer-Verlag 1979.

[8] Liniger, W.: A criterion for A-stability of linear multistep integration formulae. Computing 3 (1968), 280 - 285.

[9] Melkes, F. and M. Zlámal: Numerical solution of nonlinear quasi-stationary magnetic fields. To appear.

[10] Nečas, J.: Les méthodes directes en théorie des équations elliptiques. Academia 1967.

[11] Wheeler, M. F.: A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations. SIAM J. Numer. Anal. 10 (1973), 723 - 759.

[12] Zlámal, M.: Finite element methods in heat conduction problems. J. R. Whiteman Ed. The mathematics of finite elements and applications vol. II. Academic Press 1976.

[13] Zlámal; M.: Galerkin-finite element solution of nonlinear evolution problems. Lecture notes, University of Maryland. To appear.

[14] Zlámal, M.: Finite element solution of nonlinear quasi-stationary magnetic fields. To appear.