ON SOME SEMILINEAR VOLTERRA DIFFUSION EQUATIONS ARISING IN ECOLOGY

Yoshio Yamada

Nagoya , Japan

1. Introduction. The purpose of this lecture is to study the asymptotic behavior of solutions for a certain class of semilinear diffusion equations with memory effects. Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. We consider equations of the form

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t) + u(x,t)(a - bu(x,t) - \int_{-\infty}^{t} k(t-s)u(x,s)ds), \quad x \in \Omega,\ t > 0, \tag{1.1}$$

where a and b are non-negative constants and k is a non-negative smooth function satisfying $k,\ tk \in L^1(0,\infty)$. Equations of the form (1.1) often arise in ecology and describe the evolution of the population density of a species living in Ω. The Volterra integral in (1.1) means that past history affects the present state of the population. (For the derivation of this model, see e.g. Volterra [7].) We treat (1.1) as the initial boundary value problem with the homogeneous Neumann condition

$$\frac{\partial u}{\partial n}(x,t) = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{1.2}$$

and the initial condition

$$u(x,\tau) = \phi(x,\tau), \qquad x \in \Omega,\ \tau \leqq 0, \tag{1.3}$$

where ϕ is a given non-negative function ($\not\equiv 0$). We assume the smoothness of $\phi(x,\tau)$ in x and τ for the sake of simplicity.

Recently, asymptotic stability properties for semilinear diffusion equations with memory effects have been studied by several authors ([2],[3],[4]). Especially, Schiaffino [3] has obtained an interesting result for (1.1)-(1.3). Roughly speaking, his result says that, if k satisfies $\int_0^\infty k(t)dt \equiv \alpha < b$, then every positive solution converges to $u^* \equiv a/(b+\alpha)$ uniformly for $x \in \Omega$ as $t \to \infty$.

Our main interests lie in the following two points. The first one is to extend Schiaffino's result to give more general conditions for the asymptotic stability of the equilibrium state $u = u^*$. The second one is to study some effects which the time-delay has upon the asymptotic stability.

2. Some preliminaries. Let $p > n/2$. To treat (1.1)-(1.3) in $L^p(\Omega)$ with norm $\|\cdot\|_p$, we introduce a closed linear operator A defined by

$$Au = -\Delta u \quad \text{for} \quad u \in D(A) = \{u \in W^{2,p}(\Omega);\ \frac{\partial u}{\partial n} = 0 \ \text{on}\ \partial\Omega\}.$$

It is well known that $-A$ generates an analytic semi-group $\{e^{-tA}\}_{t\geqq 0}$ of bounded linear operators in $L^p(\Omega)$. Observe that fractional powers of A satisfy the following continuous inclusion relations (see, e.g., Henry [1])

$$D(A^\alpha) \hookrightarrow C(\Omega) \qquad \text{if} \quad n/2p < \alpha \leq 1,$$

where $D(A^\alpha)$ is equipped with the graph norm $|||u|||_{p,\alpha} = ||u||_p + ||Au||_p$.

In the usual manner, one can show the existence of a local solution for (1.1)-(1.3) by reducing it to the integral equation represented in terms of $\{e^{-tA}\}$. Moreover, by the comparison theorem, the solution is non-negative. Hence, another application of the comparison theorem enables us to conclude that (1.1)-(1.3) has a unique non-negative solution which exists for all $t \geq 0$.

3. Asymptotic stability. In this section we shall give some asymptotic stability results for (1.1)-(1.3), whose proofs can be found in [5],[6].

3.1. Global stability in the case $b > 0$. We first note

Theorem 1. The solution u of (1.1)-(1.3) satisfies

$$0 \leq u(x,t) \leq \max\{a/b, \sup_{x\in\Omega} |\phi(x,0)|\} \quad \text{for} \quad x \in \Omega \quad \text{and} \quad t \geq 0.$$

For $k \in L^1(0,\infty)$, we define its Laplace transform $\hat{k}$ by

$$\hat{k}(\lambda) = \int_0^\infty e^{-\lambda t} k(t)dt \qquad \text{for} \quad \text{Re}\,\lambda \geq 0.$$

It is said that k is a positive kernel (strongly positive kernel) if $\hat{k}(i\eta) \geq 0$ for every $\eta \in R^1$ ($\hat{k}(i\eta) \geq \gamma/(1+\eta^2)$ for every $\eta \in R^1$ with some $\gamma > 0$).

Our asymptotic stability result is

Theorem 2. If $b + \text{Re}\,\hat{k}(i\eta) > 0$ *for* $\eta \in R^1$, *then*

$$\lim_{t\to\infty} u(x,t) = u^* \left(\equiv \frac{a}{b+\alpha}\right) \quad \text{uniformly for} \quad x \in \Omega.$$

This theorem can be proved by the energy method with use of the following Liapunov functional

$$E(u) = \int_\Omega \{u(x) - u^* - u^* \log \frac{u(x)}{u^*}\}dx.$$

Theorem 2 seems to give the best possible condition for the global asymptotic stability of the equilibrium state $u = u^*$ (see Section 5).

3.2. Global stability in the case $b = 0$. In this case we require some additional assumptions to get the stability result.

Theorem 3. Let k be a positive kernel. If $u^* \int_0^\infty tk(t)dt < 1$, *then the solution u of (1.1)-(1.3) satisfies*

$$0 \leq u(x,t) \leq M \qquad \text{for} \quad x \in \Omega \quad \text{and} \quad t \geq 0,$$

with some $M > 0$.

Moreover, if k is a strongly positive kernel, then

$$\lim_{t\to\infty} u(x,t) = u^* \left(\equiv \frac{a}{\alpha}\right) \quad \text{uniformly for} \quad x \in \Omega.$$

3.3. Local stability. So far, we have discussed global stability. In order to study local stability of an equilibrium state for a nonlinear equation, it is usual to carry out the linearization procedure about that state. In our case, the linearization about u^* is

$$(3.1)\qquad \begin{cases} \dfrac{\partial v}{\partial t} = \Delta v - u^*(bv + \displaystyle\int_0^t k(t-s)v(s)ds), & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega,\ t > 0. \end{cases}$$

We shall assume

(A) For every $\operatorname{Re}\lambda \geq 0$, the "characteristic problem"

$$(3.2)\qquad \lambda w - \Delta w + u^*(b + \hat{k}(\lambda))w = 0 \quad \text{in } \Omega, \quad \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

has no non-trivial solutions.

(We say that λ satisfies the "characteristic equation" associated with (3.1) if (3.2) has a solution $w \not\equiv 0$.)

Theorem 4. Assume (A) and fix $n/2p < \alpha < 1$. For any $0 < \varepsilon < \varepsilon_0$ with some ε_0, there exists a positive number $\delta(\varepsilon)$ such that, if $\sup_{\tau\leq 0} |||\phi(\tau)-u^*|||_{p,\alpha} \leq \delta(\varepsilon)$, then the solution u of (1.1)-(1.3) satisfies

$$|||u(t) - u^*|||_{p,\alpha} \leq \varepsilon \quad \text{for all } t \geq 0,$$

and

$$\lim_{t\to\infty} |||u(t) - u^*|||_{p,\beta} = 0 \quad \text{for every } 0 \leq \beta < \alpha.$$

4. Hopf bifurcation. When the stability condition (A) is violated, what will become of the asymptotic behavior ? To study this situation, we regard one of $a, b, c, \alpha, \cdots$ as a parameter and denote it by γ. Suppose that $\lambda(\gamma)$ is a simple "characteristic root" of (3.2); thus (3.2) has a non-trivial solution. Our assumption is

(B) $\lambda(\gamma_0) = i\omega_0$ with $\operatorname{Re}\lambda'(\gamma_0) \not\equiv 0$ and $ni\omega_0$ $(n = 2,3,\cdots)$ does not satisfy the characteristic equation associated with (3.1) for $\gamma = \gamma_0$. Moreover, $u^*(\gamma_0)\hat{k}_\lambda(i\omega_0;\gamma_0) \not\equiv -1$.

Then we can show

Theorem 5. There exists a one-parameter family $(\gamma(\varepsilon), \omega(\varepsilon), u(x,s;\varepsilon))$ $(\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ with some $\varepsilon_0 > 0)$ such that

(i) $\gamma(0) = \gamma_0$, $\omega(0) = \omega_0$, $u(x,s;0) = u^*(\gamma_0)$,

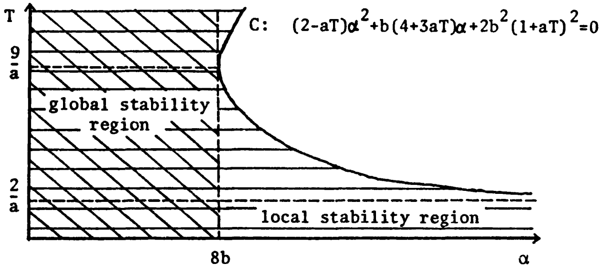
(ii) $u(x,s;\varepsilon)$ is 2π-periodic in s,

(iii) $(\gamma(\varepsilon), u(x,\omega(\varepsilon)t;\varepsilon))$ is a solution of (1.1)-(1.3).

5. Some remarks. We shall explain the preceding results by choosing a special kernel $k(t) = \alpha t \exp(-t/T)/T^2$. This kernel function takes its maximum value at $t = T$. Since $\hat{k}(\lambda) = \alpha/(1+\lambda T)^2$, the equilibrium state $u = u^*$ is globally asymptotically stable if $\alpha < 8b$ (Theorem 2). After some calculations, we see that (A) is equivalent to

$$(2-aT)\alpha^2 + b(4+3aT)\alpha + 2b^2(1+aT)^2 > 0,$$

which assures the local asymptotic stability of u^* (Theorem 4). The stability region of u^* is indicated as follows.



When (α,T) crosses the curve C, a pair of characteristic roots of (3.2) cross the imaginary axis. Hence this is the case to which Theorem 5 can be applied; we can show that non-constant periodic solutions bifurcate.

Finally, we remark that our theory developed here is extended to the study of stability for semilinear Volterra diffusion systems (see [6]).

References

[1] D. Henry; "Geometric Theory of Semilinear Parabolic Equations", Lecture Notes in Math. Vol. 840, Springer-Verlag, 1981.

[2] M. A. Pozio; Behaviour of solutions of some abstract functional differential equations and application to predator-prey dynamics, Nonlinear Analysis, Theory, Methods & Applications 4 (1980), 917-938.

[3] A. Schiaffino; On a diffusion Volterra equation, Nonlinear Analysis, Theory, Methods & Applications 3 (1979), 595-600.

[4] A. Schiaffino and A. Tesei; Global attractivity for diffusion delay logistic equations, "Nonlinear Differential Equations, Invariance, Stability and Bifurcation", edited by P. de Mottoni and L. Salvadori, Academic Press, 1981.

[5] Y. Yamada; On a certain class of semilinear Volterra diffusion equations, to appear in J. Math. Anal. Appl.

[6] Y. Yamada; Asymptotic stability for some systems of semilinear Volterra diffusion equations, preprint.

[7] V. Volterra; Leçons sur la théorie mathématique de la lutte pour la vie, Gauthier-Villars, 1931.