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In: Michal Greguš (ed.): Equadiff 5, Proceedings of the Fifth Czechoslovak Conference on Differential Equations and Their Applications held in Bratislava, August 24-28, 1981. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982. Teubner-Texte zur Mathematik, Bd. 47. pp. 284--287.

Persistent URL: http://dml.cz/dmlcz/702307

HOMOGENIZATION OF DIFFERENTIAL OPERATORS

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In last years the theory of homogenization and G-convergence of differential operators was developed (see [I]-[4]and others). Various problems in the mechanics of strongly inhomogeneous media lead to the necessity of constructing homogenized models for these media. In many cases the physical processes in these media can be described by partial differential equations with abruptly varying coefficients. Such questions arise in the theory of elasticity, of heterogeneous media and composite materials, of filtration and in many other branches of physics and mechanics. A direct numerical solution of such problems is very difficult even with the aid of modern computers. It leads to a problem of constructing a so called homogenized differential equation (it often has constant coefficients) and the basic requirement that one has to impose on the homogenized equation is the proximity of the solutions of the corresponding boundary-value problems for the original equations and the homogenized equation. This leads to a concept of G-convergence of differential operators.

The theory of homogenization and G-convergence for elliptic operators is described in [2]. Here we state some results on homogenization of parabolic equations which are obtained jointly with V.V. Zhikov and S.M. Kozlov.

Let us consider a parabolic operator of the form

$$\mathcal{P} = \frac{\partial}{\partial t} + \sum_{|\alpha|,|\beta| \le m} (-1)^{|\alpha|} \mathcal{D}^{\alpha}\left(a_{\alpha\beta}(x,t)\mathcal{D}^{\beta}\right), \tag{I}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ are multi-indices, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\mathcal{D}^{\alpha} = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$. Suppose that $a_{\alpha\beta}(x,t)$ are real measurable functions in $R^{n+1}_{x,t}$ and for any $t \geq 0$

$$\sup_{R^n_x} |a_{\alpha\beta}(x,t)| \le M, \quad |\alpha|, |\beta| \le m, \tag{2}$$

$$\int_{R_x^n} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x,t) \mathcal{D}^\beta u \, \mathcal{D}^\alpha u \, dx \geq \lambda_0 \int_{R_x^n} \sum_{|\alpha|=m} |\mathcal{D}^\alpha u|^2 dx, \quad (3)$$

where λ_0, M are positive constants, $u \in C_0^\infty(\Omega)$. We denote by $P(\lambda_0, M)$ the class of operators of the form (I) for which (2) and (3) are valid. Operators from $P(\lambda_0, M)$ belong to a more general class of operators for which the theory of the strong G-convergence is described in [4].

Let Ω be a bounded domain in R_x^n, $Q = \Omega \times [0,T]$, $T = const > 0$. Let us denote by $C^\infty(\omega, \gamma)$ the set of functions infinitely differentiable in a neighbourhood of ω and equal to zero on γ. We denote by $\overset{\circ}{H}{}^m(\Omega)$ the completion of $C^\infty(\Omega, \partial\Omega)$ in the norm

$$\|u\|_{\overset{\circ}{H}{}^m(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \leq m} |\mathcal{D}^\alpha u|^2 dx \right)^{\frac{1}{2}}.$$

We set $(\overset{\circ}{H}{}^m(\Omega))^* = H^{-m}(\Omega)$. Denote by $\mathcal{V}, \mathcal{V}', \mathcal{W}$ the completion of $C^\infty(Q, \partial\Omega \times [0,T])$ respectively in the norms

$$\|u\|_{\mathcal{V}} = \left(\int_0^T \|u\|^2_{\overset{\circ}{H}{}^m(\Omega)} dt \right)^{\frac{1}{2}}, \quad \|u\|_{\mathcal{V}'} = \left(\int_0^T \|u\|^2_{H^{-m}(\Omega)} dt \right)^{\frac{1}{2}},$$

$$\|u\|_{\mathcal{W}} = \left(\|u\|^2_{\mathcal{V}} + \left\|\frac{\partial u}{\partial t}\right\|^2_{\mathcal{V}'} \right)^{\frac{1}{2}}.$$

It is proved that if $\mathcal{P} \in P(\lambda_0, M)$, then the problem $\mathcal{P}u = f$, $u|_{t=0} = \psi$, $f \in \mathcal{V}'$, $\psi \in L^2(\Omega)$, has a unique solution $u \in \mathcal{W}$.

Let $\mathcal{P}_\varepsilon$ be a family of operators, $0 < \varepsilon \leq 1$, $\mathcal{P}_\varepsilon \in P(\lambda_0, M)$, and $\mathcal{P}_\varepsilon u_\varepsilon = f$, $u_\varepsilon|_{t=0} = \psi$, $f \in \mathcal{V}'$, $\psi \in L^2(\Omega)$, $u_\varepsilon \in \mathcal{W}$. Suppose $\widehat{\mathcal{P}} \in P(\hat{\lambda}_0, \hat{M})$, $\widehat{\mathcal{P}}u = f$, $u|_{t=0} = \psi$, $u \in \mathcal{W}$. We say that strongly G-converges to the operator $\widehat{\mathcal{P}}$ as $\varepsilon\to 0$ ($\mathcal{P}_\varepsilon \xrightarrow{G} \widehat{\mathcal{P}}$), if $u_\varepsilon \to u$ weakly in $\mathcal{W}$ and

$$\sum_{|\beta|\le m} a^\varepsilon_{\alpha\beta}(x,t)\mathcal{D}^\beta u_\varepsilon \longrightarrow \sum_{|\beta|\le m} \widehat{a}_{\alpha\beta}(x,t)\mathcal{D}^\beta u, \quad |\alpha|\le m,$$

weakly in $L^2(Q)$ as $\varepsilon\to 0$ for any $f\in\mathcal{V}'$, $\psi\in L_2(\Omega)$, where $a^\varepsilon_{\alpha\beta}$ are coefficients of $\mathcal{P}_\varepsilon$, $\widehat{a}_{\alpha\beta}$ are coefficients of $\widehat{\mathcal{P}}$, $|\alpha|,|\beta|\le m$.

We consider a family of operators $\mathcal{P}_\varepsilon$ of the form

$$\mathcal{P}_\varepsilon = \frac{\partial}{\partial t} + \sum_{|\alpha|,|\beta|\le m} (-1)^{|\alpha|}\mathcal{D}^\alpha\left(a_{\alpha\beta}(\varepsilon^{-\ell_1}x, \varepsilon^{-\ell_2}t)\mathcal{D}^\beta\right), \tag{4}$$

where $\mathcal{P}_1$ is an operator from the class $P(\lambda_0, M)$; $\ell_1, \ell_2 = const \ge 0$, $\ell_1+\ell_2>0$, $\varepsilon = const$, $\varepsilon>0$. It is easy to see that $\mathcal{P}_\varepsilon \in P(\lambda_0, M)$.

We say that $\mathcal{P}_\varepsilon$ admits homogenization as $\varepsilon\to 0$, if $\mathcal{P}_\varepsilon \xrightarrow{G} \widehat{\mathcal{P}}$ and $\widehat{\mathcal{P}}\in P(\widehat{\lambda}_0, \widehat{M})$. We consider the cases: I) $\ell_2 = 2m\ell_1$; 2) $\ell_2 > 2m\ell_1$; 3) $\ell_2 < 2m\ell_1$; 4) $\ell_2 = 0$; 5) $\ell_1 = 0$.

Theorem I. Suppose that $\mathcal{P}_1$ given by (4) for $\varepsilon = 1$ belongs to $P(\lambda_0, M)$. Then if $a_{\alpha\beta}(x,t)$, $|\alpha|,|\beta|\le m$, are almost-periodic functions in $R^{n+1}_{x,t}$, the family of operators $\mathcal{P}_\varepsilon$, given by (4), in cases I), 2), 3) admits homogenization as $\varepsilon\to 0$ and the operator $\widehat{\mathcal{P}}$ has constant coefficients. In cases 2) and 3) operator $\widehat{\mathcal{P}}$ does not depend on ℓ_1, ℓ_2. If $a_{\alpha\beta}(x,t)$ are uniformly continuous in $R^n_x\times[0,T]$ and almost-periodic with respect to x, then in case 4) ($\ell_2=0$) the family $\mathcal{P}_\varepsilon$ admits homogenization as $\varepsilon\to 0$ and coefficients $\widehat{a}_{\alpha\beta}$ of $\widehat{\mathcal{P}}$ do not depend on x, $\widehat{\mathcal{P}}\in P(\lambda_0, \widehat{M})$. If $a_{\alpha\beta}(x,t)$ are uniformly continuous in R^{n+1} and for any $x\in\Omega$, any α,β with $|\alpha|\le m$, $|\beta|\le m$ there exist $a^*_{\alpha\beta}(x)$ defined by (6), then $\mathcal{P}_\varepsilon$ admits homogenization for $\ell_1 = 0$ and $\hat{a}_{\alpha\beta} = a^*_{\alpha\beta}(x)$.

Theorem I is also valid for a larger class of operators $\mathcal{P}_\varepsilon$ (see [5]). It is proved also that under conditions of theorem 1

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \int_\Omega |u_\varepsilon(x,t) - u(x,t)|^2 dx = 0.$$

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