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THE PERIODIC BOUNDARY VALUE PROBLEM FOR SOME

SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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We shortly describe recent results obtained by J.R. WARD and the author about the existence of solutions for the periodic boundary value problem

$$
\begin{aligned}
&x'' + h(x)x' + f(t,x) = e(t) \\
&x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0
\end{aligned} \tag{1}
$$

where $h : \mathbb{R} \to \mathbb{R}$ is continuous, $e \in L^1(0,2\pi)$, f satisfies the Caratheodory conditions and asymptotic conditions of the form

$$\gamma(t) \leq \liminf_{|x| \to \infty} x^{-1}f(t,x), \tag{2}$$

$$\limsup_{|x| \to \infty} x^{-1}f(t,x) \leq \Gamma(t), \tag{3}$$

where $\gamma \in L^1(I)$, $\Gamma \in L^1(I)$, $I = [0,2\pi]$ and the relations (2) are supposed to hold uniformly a.e. on I. If $g : I \to \mathbb{R}$ and if $a \in \mathbb{R}$, we introduce the notation

$$g(t) \lesssim a$$

on I if $g(t) \leq a$ on I and $g(t) < a$ on a subset of I positive measure.

In [3], we proved that (1) is solvable for every h and every e if (3) holds and

$$\Gamma(t) \lesssim 0 \text{ on } I, \tag{4}$$

i.e. if $\Gamma(t) \leq 0$ a.e on I and $\bar{\Gamma} =: (2\pi)^{-1} \int_I \Gamma(t)dt < 0$.
Notice that 0 is the smallest value of μ such that the problem

$$
\begin{aligned}
&x'' + \mu x = 0 \\
&x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0
\end{aligned} \tag{5}
$$

has non-trivial solutions, the other ones being $k^2 (k \in \mathbb{N}^*)$.
The example of Amaral and Pera [1]

$$
\begin{aligned}
&x'' + \frac{\sin t}{a+\sin t} x = e(t), \quad a > 1, \\
&x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0
\end{aligned} \tag{6}
$$

for which $(2\pi)^{-1} \int_I \frac{\sin t}{a+\sin t} dt < 0$ but for which the homogeneous problem has the non-trivial solutions $c(a+\sin t)$ $(c \in \mathbb{R})$, so that (6) cannot be solved for every $e \in L^1(I)$, shows that the first condition in (4) cannot be completely dropped.

GOSSEZ [2] however has proved that (1) is solvable for every h and e if

$$(7) \qquad \overline{\Gamma^+} < 3/4\ \pi^2,\ \overline{\Gamma^+} - [\,1-(4\pi^2/3)\overline{\Gamma^+}\,]^{1/2}\ \overline{\Gamma^-} < 0$$

where $\Gamma^+ = \max(\Gamma,0)$, $\Gamma^- = \max(-\Gamma,0)$, and (7) obviously reduces to (4) when $\Gamma^+ = 0$, i.e when $\Gamma(t) \leq 0$ a.e on I.

Existence results also hold for (1) when (2) and (3) hold and γ and Γ are related to two consecutive squares of integers. Because of the simplest nature of the non-trivial solutions of (5) when $\mu = 0$, better results are obtained for γ and Γ respectively related to 0 and 1 with respect to the other cases $(k-1)^2$ and $k^2 (k \geq 2)$. They are summarized in the following result extending previous ones of LAZER, REISSIG, CHANG, MARTELLI, GUPTA, AMARAL and PERA, and the author.

THEOREM. *Assume that conditions* (2) *and* (3) *hold and that* γ, Γ, h *and* e *satisfy one of the following conditions:*

1. h *is constant, there is an integer* $k \geq 2$ *such that*

$$(k-1)^2 \lesssim \gamma(t) \leq \Gamma(t) \lesssim k^2 \text{ on } I,$$

and e *is arbitrary.*

2. h *is constant,* $\bar{\gamma} = 0$, $\gamma \not\equiv 0$, $\Gamma(t) \lesssim 1$ *on* I *and* e *is arbitrary.*

3. h *is arbitrary,* $\bar{\gamma} > 0$, $\Gamma(t) \lesssim 1$ *on* I *and* e *is arbitrary.*

4. h *is arbitrary,* $0 = \bar{\gamma} < \bar{\Gamma}$, $\Gamma(t) \lesssim 1$ *on* I,

$$(8) \qquad (2\pi)^{-1} \int_I f(t,x(t))dt \geq A \text{ (resp. } (2\pi)^{-1} \int_I f(t,x(t))dt \leq a)$$

when $x \in AC^1_{2\pi}$ *and* $\min_{t\in I} x(t) \geq R$ (*resp.* $\max_{t\in I} x(t) \leq r$), *for some* $a \leq A$ *and* $r < 0 < R$, *where* $AC^1_{2\pi} = \{x : I \to \mathbb{R} : x$ *and* x' *are absolutely continuous on* I *and* $x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0\}$, *and finally*

$$a \leq \bar{e} \leq A.$$

Then problem (1) *has at least one solution.*

This theorem is proved by degree arguments. The required a priori bounds for the use of the method of proof are deduced from the obtention of a priori bounds for problems of the form

$$x'' + h(x)x' + p(t)x = q(t)$$

where $p, q \in L^1(I)$ and where $\gamma(t)-\varepsilon \leq p(t) \leq \Gamma(t)+\varepsilon$ on I for some $\varepsilon > 0$ sufficiently small. Those a priori bounds are themselve deduced from coercivity properties on the Sobolev space $H^1(I)$ of some quadratic forms associated to γ and Γ. We refer to [4], [5] and [6] for the details and proofs.

<u>**Examples**</u>.

1. If $I = I^1 \cup I^2$, with $0 < \text{meas } I^1 < 2\pi$ and if, for some $k \in \mathbb{N}^*$, $k \geq 2$, we define $p : I \to \mathbb{R}$ by $p(t) = (k-1)^2$ for $t \in I^1$ and $p(t) = k^2$ for $t \in I^2$, then for each $q \in L^1(I)$, the problem

$$(9) \qquad \begin{aligned} x'' + p(t)x &= q(t) \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) &= 0 \end{aligned}$$

has by part 1 of the above theorem, a (unique) solution. So we have a *nonresonance* situation although the coefficient $p(t)$ only takes *resonant* values $(k-1)^2$ and k^2; but it takes two and not only one resonant value !

2. If $I = I^1 \cup I^2 \cup I^3$ with meas $I^i < 2\pi$ $(i = 1,2)$ $(I = 1,2,3)$ and if we define $p : I \to \mathbb{R}$ by $p(t) = 1$ for $t \in I^1$, $p(t) = 0$ for $t \in I^2$, $p(t) = -\nu$ for $t \in I^3$ and some $\nu > 0$, then the problem (9) will have , by part 2 or 3 of the above theorem, a (unique) solution for each $q \in L^1(I)$ if ν meas $I^3 \leq$ meas I^1. It is again a nonresonant situation and the restriction is only on the measure of the set I^3 on which p takes at the *nonresonant* value $-\nu$!

3. If, for some integer $m \geq 1$, and some $0 < \varepsilon \leq 1$, $p(t) = \varepsilon \sin mt$, then, by part 2 of the theorem, the problem (9) has a (unique) solution for each $q \in L^1(I)$. Thus it is a nonresonant problem. Notice that in this case the *averaged* homogeneous problem

$$x'' = 0, \ x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$$

is resonant !

4. Interesting special cases under which condition (8) is satisfied are either $f(t,x) \geq B(t)$ for a.e. $t \in I$, all $x \geq R$ and some $B \in L^1(I)$ and $f(t,x) \leq b(t)$ for a.e. $t \in I$, all $x \leq r$ and some $b \in L^1(I)$ with $\bar{b} < \bar{B}$, or $f(t,x) \geq \delta_+(t)$ (resp. $f(t,x) \leq \delta_-(t)$) for a.e. $t \in I$, all $x \geq 0$, (resp. $x \leq 0$), and some $\delta_+ \in L^1(I)$, and setting $f^+(t) = \liminf_{x \to +\infty} f(t,x)$, $f^-(t) = \limsup_{x \to -\infty} f(t,x)$, $\overline{f^-} < \overline{f^+}$. Then existence is insured in the first case for all $e \in L^1(I)$ with $\bar{b} < \bar{e} < \bar{B}$ and in the second case for all $e \in L^1(I)$ with $\overline{f^-} < \bar{e} < \overline{f^+}$.

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