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FREE VIBRATIONS FOR THE EQUATION $u_{tt}-u_{xx}+f(u) = 0$ WITH f SUBLINEAR

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Summary: The assumptions on a function f are found under which the equation $u_{tt}-u_{xx}+f(u) = 0$ with the boundary conditions $u(t,0) = u(t,\pi) = 0$ has a nontrivial 2π-periodic solution.

1. Notation.

The symbol $\int v$ denotes the integral of a function v over $(0,2\pi) \times (0,\pi)$. By L_p, $1 \leq p < \infty$ (or L_∞), we denote the space of real-valued measurable functions u on $R \times (0,\pi)$, 2π-periodic in the first variable and satisfying $|u|_p = \left(\int |u|^p\right)^{1/p} < \infty$ (or $|u|_\infty = \sup \operatorname{ess} |u(t,x)| < \infty$, respectively).

The functions e_{jk} are defined on $R \times (0,\pi)$ by

$$e_{jk}(t,x) = \begin{cases} \frac{\sqrt{2}}{\pi} \cos jt \sin kx & \text{for } j, k \in N , \\ \frac{1}{\pi} \sin kx & \text{for } j = 0 , \ k \in N , \\ \frac{\sqrt{2}}{\pi} \sin jt \sin kx & \text{for } -j, k \in N . \end{cases}$$

For $u \in L_1$ we put

$$a_{jk}(u) = \int u e_{jk} .$$

2. Weak 2π-periodic solutions of the wave equation.

Let f be a real-valued function on R. A function $u \in L_1$ is called a (weak 2π-periodic) solution to the problem

$$(1) \qquad u_{tt} - u_{xx} + f(u) = 0 , \quad u(t,0) = u(t,\pi) = 0 ,$$

if the composed function $f(u)$ belongs to L_1 and

$$(j^2 - k^2) a_{jk}(u) = a_{jk}(f(u))$$

for any j, k.

In the paper [1] the existence of a nontrivial solution to (1) with f of the form

$$(2) \qquad f(u) = |u|^\alpha \operatorname{sgn}(u) \quad (0 < \alpha < 1)$$

is established. In the paper [2] the existence of nontrivial T-periodic solutions (T sufficiently large) to (1) is proved for a rather general class of sublinear functions f.

3. Formulation of main results.

Let us denote by S (or S') the set of all functions f which fulfil the following assumptions (S1) - (S4) (or (S1) - (S5), respectively):

(S1) $f \in C(R,R)$, odd, increasing;

(S2) f is continuously differentiable on $R \setminus \{0\}$ and

$$f(u)u \geq f'(u)u^2 \quad \text{for} \quad u \neq 0 ;$$

(S3) there exist constants $c_1 > 0$ and $\delta \in (0,1)$ such that

$$f(u) \geq c_1 u^\delta \qquad \text{for} \quad u > 0 ;$$

(S4) there exist constants $c_2, c_3 > 0$ and $p > 2$ such that

$$\int_0^u f(s)ds - \frac{1}{2} uf(u) \geq c_2|f(u)|^p - c_3 \quad \text{for} \quad u \in R ;$$

(S5) the function $u \to uf(u)$ is convex.

Let us note that any function f of the form (2) belongs to S' and that $f_1, f_2 \in S'$ and $a, b > 0$ implies $af_1 + bf_2 \in S'$.

THEOREM 1. For any $f \in S$ there exists a nontrivial solution $u \in L_\infty$ to the problem (1).

THEOREM 2. Let $f \in S'$ and let us denote $F(u) = \int_0^u f(s)ds$ for $u \in$ $\in R$. Then there exists a sequence $\{u_n;\ n \in N\}$ of solutions to (1), such that $u_n \in L_\infty$ $(n \in N)$ and $\{ \int (F(u_n) - \frac{1}{2}u_n f(u_n));\ n \in N \}$ forms a decreasing sequence of positive reals with 0 as a limit point.

4. Sketch of proofs.

a) Let $f \in S$. First we shall seek solutions of the "modified" problem

$$(1_\varepsilon) \qquad u_{tt} - u_{xx} + f_\varepsilon(u) = 0 , \quad u(t,0) = u(t,\pi) = 0 ,$$

where $f_\varepsilon(u) = f(u) + \varepsilon|u|^{1/p-1}\,\mathrm{sgn}(u)$ (and p is the same as in (S4)).

b) Approximate solutions for (1_ε) will be obtained as critical points of functionals $g_{n,\varepsilon}$, defined on $H_n = \text{lin}\{e_{jk};\ |j| \leq n,\ k \leq n\}$ by

$$g_{n,\varepsilon}(u) = -\frac{1}{2}\int(u_t^2-u_x^2) + \int F_\varepsilon(u) ,$$

where $F_\varepsilon(u) = \int_0^u f_\varepsilon(s)ds$.

c) The following assertion plays a fundamental role: For any $a > 0$ there exists $k(a) \in (0,a)$ such that for a sufficiently large n and $\varepsilon \in (0,1)$ there exists a critical point $u_{n,\varepsilon}$ of $g_{n,\varepsilon}$ with $g_{n,\varepsilon}(u_{n,\varepsilon}) = \int\{F_\varepsilon(u_{n,\varepsilon}) - \frac{1}{2}u_{n,\varepsilon}f_\varepsilon(u_{n,\varepsilon})\} \in [k(a),a]$.

In order to obtain those appropriate approximate solutions, the Ljusternik-Schnirelmann theory is used.

d) Let $\varepsilon \in (0,1)$ be fixed. Then it may be shown (by a monotonicity argument) that a certain subsequence of $\{u_{n,\varepsilon};\ n \in N\}$ converges weakly in $L_{p'}$ (where p' is conjugate to p) to a solution $u_\varepsilon \in L_{p'}$ of (1_ε) and that, moreover, $\int u_\varepsilon f_\varepsilon(u_\varepsilon) \geq 2k(a) > 0$ (i.e. that u_ε is a nontrivial solution).

e) As u_ε solves (1_ε), the relation

$$\int_0^\pi \{f_\varepsilon(u_\varepsilon(t-x,x)) - f_\varepsilon(u_\varepsilon(t+x,x))\}dx = 0$$

is valid for a.e. t . By using this fact it may be shown that u_ε belong to L_∞ and are bounded in L_∞ uniformly with respect to $\varepsilon \in (0,1)$.

f) By making use of the above assertion it is possible to obtain by the limiting process for $\varepsilon \to 0$ (again mainly by a monotonicity argument) a solution $u \in L_\infty$ to the problem (1) with $\int uf(u) \geq$ $\geq 2k(a) > 0$, which proves Theorem 1.

g) If $f \in S'$ then it may be shown that the solution u obtained by the above procedure satisfies $\int\{F(u) - \frac{1}{2}uf(u)\} \in [k(a),a]$, which easily implies the validity of Theorem 2.

References

[1] H. Brézis, J.-M. Coron, L. Nirenberg: Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz. Comon. Pure Appl. Mat. 33(1980), 667-689.

[2] H. Brézis, J.-M. Coron: Periodic solutions of nonlinear wave equations and Hamiltonian systems (preprint).