Milan Kučera

Bifurcation problems for variational inequalities

In: Michal Greguš (ed.): Equadiff 5, Proceedings of the Fifth Czechoslovak Conference on Differential Equations and Their Applications held in Bratislava, August 24-28, 1981. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982. Teubner-Texte zur Mathematik, Bd. 47. pp. 209--211.

Persistent URL: http://dml.cz/dmlcz/702290

BIFURCATION PROBLEMS FOR VARIATIONAL INEQUALITIES

Milan Kučera
Praha, Czechoslovakia

We shall consider a real Hilbert space H and a closed convex cone K in H with vertex at the origin. The inner product and the corresponding norm in H are denoted by $\langle .,. \rangle$ and $||.||$, respectively. We shall suppose that A is a linear completely continuous operator in H, $N : \mathbb{R} \times H \to H$ a nonlinear completely continuous mapping such that

$$\lim_{\|v\| \to 0} \frac{N(\mu,v)}{\|v\|} = 0 \quad \text{uniformly on bounded } \mu\text{-intervals.}$$

We shall deal with a bifurcation problem for the variational inequality

(I) $\qquad u \in K$,

(II) $\qquad \langle u-\mu Au+N(\mu,u),v-u \rangle \geq 0$ for all $v \in K$.

In connection with the variational approach it is natural to investigate the variational inequality

(II') $\qquad \langle f'(u)-\mu g'(u),v-u \rangle \geq 0$ for all $v \in K$,

where f, g are functionals on H, f', g' their Frèchet derivatives. A point $[\mu_0,0]$ is said to be a bifurcation point of (I), (II) if every its neighbourhood in $\mathbb{R} \times H$ contains a couple μ, u satisfying (I), (II) with $||u|| \neq 0$. Analogously for (I), (II'). Our aim is to explain briefly some results concerning higher bifurcation points (greater then the first one).

<u>1. VARIATIONAL APPROACH.</u> First, we must mention the results of E. Miersemann [7], [8] based on a modification of Krasnoselskii's sup--min principle. A closed subspace $H_1 \subset K$ is considered and it is supposed that $\tilde{\mu}_n < \mu_{n+1}$ (n fixed), where μ_n and $\tilde{\mu}_n$ denote the n-th characteristic value of the linear eigenvalue problem

$$u \in H , \quad f''(0)(u,v) - \mu g''(0)(u,v) = 0 \quad \text{for all} \quad v \in H$$

and

$$u \in H_1 , \quad f''(0)(u,v) - \mu g''(0)(u,v) = 0 \quad \text{for all} \quad v \in H_1 ,$$

respectively. Under certain assumptions about f, g (which are fulfilled for example if $f(u) = \frac{1}{2} ||u||^2$, g is weakly continuous, $g' = A+N$ with A, N as above but N independent of μ and uniformly continuous on bounded sets), the existence of a bifurcation point $\mu_{b,n} \in \langle\mu_n,\tilde{\mu}_n\rangle$ of (I), (II') is proved. If there is an eigenvector corresponding to μ_n not lying in K, then even $\mu_{b,n} \in (\mu_n,\tilde{\mu}_n\rangle$. In the examples, this ensures the existence of a finite number (depending on the relations between μ_n, $\tilde{\mu}_n$) of bifurcation points.

Let us remark that in [2] a modification of the Ljusternik-Schnirelmann theory was used for the problem with the penalty corresponding to (I), (II'). This method gives formally infinitely many solutions of (I), (II') satisfying $f(u) = r (r > 0$ fixed) but only in the case when K is a halfspace it is proved that they are mutually different.

2. TOPOLOGICAL APPROACH. Now we shall explain one result obtained by a method developed in [4], [5], [6] (cf. also [3]). For simplicity we shall suppose that A is symmetric (in the nonsymmetric case the situation is formally more complicated [6]). Suppose that there exists an operator $\beta : H \to H$ (a penalty operator) which is completely continuous, monotone ($\langle\beta u-\beta v,u-v\rangle \geq 0$ for $u, v \in H$), positive homogeneous ($\beta(tu) = t\beta u$ for $t > 0$, $u \in H$) and such that $\beta u = 0$ for all $u \in K$, $\langle\beta v,v\rangle > 0$ for all $v \notin K$, $\langle\beta v,u\rangle < 0$ for all $v \notin K$, $u \in K^0$ (the interior of K). Let μ_0, μ_1 be simple characteristic values of A with the corresponding eigenvectors u_0, $u_1 \in K^0$ (the interior of K), $-u_0$, $-u_1 \notin K$. Suppose that there is no characteristic value of A in (μ_0,μ_1) having an eigenvector in K. Then there exists a bifurcation point $[\mu_b,0]$ of (I), (II) with $\mu_b \in (\mu_0,\mu_1)$. The bifurcating solutions can be obtained from the branch of solutions of the equation with the penalty. More precisely, for each $\delta > 0$ denote by C_δ the closure (in $\mathbb{R} \times H \times \mathbb{R}$) of the set of all triplets $[\mu,u,\varepsilon] \in \mathbb{R} \times H \times \mathbb{R}$ satisfying the conditions $\varepsilon \neq 0$,

(a) $$||u||^2 = \frac{\delta\varepsilon}{1+\varepsilon}$$

(b) $$u - \mu Au + \frac{\varepsilon}{1+\varepsilon} N(\mu,u) + \varepsilon\beta u = 0 .$$

Under our assumptions, there exists $\delta_0 > 0$ such that for each $\delta \in (0,\delta_0\rangle$ there is an unbounded (in ε) closed connected subset $C_{\delta,0}$ of C_δ containing $[\mu^{(0)},0,0]$, lying in (μ_0,μ_1) (in μ) and outside of K (in u) with the exception of some isolated points. If μ_n, u_n, $\varepsilon_n \in C_{\delta,0}$, $\varepsilon_n \to +\infty$, $\mu_n \to \mu(\delta)$, $v_n \to v(\delta)$ weakly, then we obtain (using a modified penalty method) that $\mu(\delta)$, $v(\delta)$ satisfy (I), (II) and it can be proved that the limiting points of $\mu(\delta)$ for $\delta \to 0+$ lie in (μ_0,μ_1).

The existence of the branch $C_{\delta,0}$ with the above mentioned properties can be proved on the basis of a global bifurcation result [1] and the fact that (a), (b) can be understood as a bifurcation equation of the usual type.

In the case $N \equiv 0$, the results of this type for multiple characteristic values μ_0, μ_1 are proved in [5].

In some cases the method gives the existence of an infinite sequence of bifurcation points of (I), (II) converging to infinity.

3. REMARKS. All mentioned results are applicable to variational inequalities describing a beam or a plate which is compressed and unilaterally supported.

We could mention a number of authors who have treated some questions connected with bifurcations of variational inequalities, but most of them either deal with problems of a different type (some symmetry assumptions about K and f, g are considered) or their interest is concentrated on the first bifurcation point only. For the references see [8], [4].

R E F E R E N C E S

[1] E. N. Dancer: On the structure of solutions of non-linear eigenvalue problems. Indiana Univ. Math. Journ., Vol. 23, No. 11 (1974), 1069-1076.

[2] M. Kučera, J. Nečas, J. Souček: The eigenvalue problem for variational inequalities and a new version of the Ljusternik--Schnirelmann theory. In "Nonlinear Analysis", Academic Press, New York-San Francisco-London 1978.

[3] M. Kučera: A new method for obtaining eigenvalues of variational inequalities of the special type. Preliminary communication. Comment. Math. Univ. Carol. 18,1 (1977), 205-210.

[4] M. Kučera: A new method for obtaining eigenvalues of variational inequalities based on bifurcation theory. Časopis pro pěstování matematiky, 104 (1979), 389-411.

[5] M. Kučera: A new method for obtaining eigenvalues of variational inequalities. Operators with multiple eigenvalues. To appear in Czechoslovak Math. Journ.

[6] M. Kučera: Bifurcation points of variational inequalities. To appear in Czechoslovak Math. Journ.

[7] E. Miersemann: Über höhere Verzweigungspunkte nichtlinearer Variationsungleichungen. Math. Nachr. 85 (1978), 195-213.

[8] E. Miersemann: Höhere Eigenwerte von Variationsungleichungen. To appear in Beiträge zur Analysis.