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CAUCHY'S PROBLEM IN THE LARGEFOR NONLINEAR HYPERBOLIC EQUATIONS AND FOR THE KORTEWEG - DE VRIES EQUATION

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In introductory part of the paper it is given the short survey of the development of the global solvability theory to the Cauchy problem in the stripe $\Pi_T = (0,T)\times R^1$ for the equation

$$u_t + (F(u))_x = 0 \ , \ u = u(t,x) \qquad (1)$$

with the initial condition

$$u(0,x) = u_0(x) \ . \qquad (2)$$

One of the basic methods of the construction of the generalized solution $u(t,x)$ to the problem (1),(2) is the "vanishing viscosity method": $u(t,x) = \lim_{\varepsilon\to+0} u^\varepsilon(t,x)$, where $u^\varepsilon(t,x)$ - is the solution of the Cauchy problem for the parabolic equation

$$u^\varepsilon_t + (F(u^\varepsilon))_x = \varepsilon\, u^\varepsilon_{xx} \ . \qquad (3)$$

This method was justified firstly in the well known paper of E.Hopf [1] in the case $F(u) = u^2/2$. The limiting process as $\varepsilon\to+0$ in the corresponding equation (3) (the Burgers's equation)

$$u^\varepsilon_t + u^\varepsilon u^\varepsilon_x = \varepsilon\, u^\varepsilon_{xx} \qquad (4)$$

leads to the notion of the generalized solution $u(t,x)$ of the equation

$$u_t + u u_x = 0 \qquad (5)$$

in the sense of the integral identity

$$\iint_{\Pi_T} (u\varphi_t + \tfrac{1}{2} u^2 \varphi_x)\, dx\, dt = 0 \quad \forall \varphi \in \overset{\circ}{C}{}^\infty(\Pi_T) \ ; \qquad (6)$$

the above solutions satisfy the entropy condition in the points of the discontinuity (along the shock waves):

$$u(t,x-0) > u(t,x+0) \ . \qquad (7)$$

The global solvability questions of the problem (1),(2) for the convex $F(u)$ were considered in the papers of O.A.Oleĭnik [2], P.D.Lax [3], A.N.Tikhonov and A.A.Samarskiĭ [4]. For the nonconvex $F(u)$ the theory of this problem in the class of bounded and measurable functions is constructed in the paper [5].

Note that the generalized [(6),(7)] -solution of the problem (5),(2) with the initial function $u_0(x) = \operatorname{sign} x - \operatorname{sign}(x-1)$ for $t \geq 2$ is the function

$$u(t,x)=\begin{cases} 0 & \text{for} \quad x \le 0 \quad \text{and} \quad x \ge \sqrt{2t}, \\ x/t & \text{for} \quad 0 < x < \sqrt{2t}. \end{cases}$$

In particular (according to the effect of the dissipation of the energy at the shock waves)

$$\int_{-\infty}^{+\infty} u^2(t,x)\,dx = \frac{2\sqrt{2}}{3}\,\frac{1}{\sqrt{t}} \to 0 \quad \text{as } t \to +\infty. \tag{8}$$

The domination of the dispersion over the dissipation in the simplest gas-dynamic model leads to the below Korteweg-de Vries's equation (KdV)

$$u_t^\mu + u^\mu u_x^\mu = -\mu\, u^\mu_{xxx}, \mu = const. \tag{9}$$

Since for the solution $u^\mu(t,x)$ of the problem (9),(2) the conservation law

$$\int_{-\infty}^{+\infty} (u^\mu(t,x))^2\,dx = \int_{-\infty}^{+\infty} u_0^2(x)\,dx \tag{10}$$

is valid then according to (8) the functions $u^\mu(t,x)$ cannot converge strongly as $\mu \to 0$ to the generalized solution $u(t,x)$ of the problem (5),(2). The first result on the weak convergence $u^\mu(t,x)$ as $\mu \to 0$ is established in the paper of P.D.Lax and C.D.Levermore [6] under the condition that $u_0(x)$ has a single maximum.

But the Cauchy problem for KdV equation with discontinuous initial data was not studied essentially. In particular uniqueness theorems are not available and the existence is proved under the strict restriction on the structure of the initial function. Different solvability questions for KdV equation were considered in [6] - [15] .

In the main part of this paper the theory of generalized solutions to the Cauchy problem for KdV equation is considered. Here the results on existence, uniqueness and regularity are formulated; these results are new and for the case of smooth initial data. They are received by the author jointly with A.V.Faminskii and they will be published in DAN USSR.

Generalized Solutions of KdV Equation

Assume that $u_0(x) \in L_2(R^1)$ and for the simplicity we shall consider KdV equation of the form

$$u_t + u u_x + u_{xxx} = 0. \tag{11}$$

<u>Definition 1</u>. A function $u(t,x) \in L_2^{loc}(\Pi_T)$ is called a <u>generalized solution</u> of the problem (11),(2) in the stripe Π_T if:

1) $\forall \varphi \in C^\infty(\Pi_T)$

$$\iint\limits_{\Pi_T} (u\varphi_t + \tfrac{1}{2}u^2\varphi_x + u\varphi_{xxx})\,dx\,dt = 0\,, \qquad (12)$$

2) there exists a set $\mathcal{E} \subset [0,T]$, $mes\,\mathcal{E}=0$, such that for $t\in[0,T]\setminus\mathcal{E}$ the function $u(t,x)$ is defined almost everywhere in R^1 and for $\forall\omega(x)\in \overset{\circ}{C}{}^{\infty}(R^1)$

$$\int_{-\infty}^{+\infty} u(t,x)\omega(x)\,dx \to \int_{-\infty}^{+\infty} u_0(x)\omega(x)\,dx, \text{ as } t\to+0,\ t\in[0,T]\setminus\mathcal{E}. \qquad (13)$$

Theorem 1(existence). Suppose that for some $\alpha>0$

$\int_0^{+\infty} x^{\frac{1}{2}+\alpha}u_0^2\,dx + \int_{-\infty}^{+\infty} u_0^2\,dx < \infty$. Then the generalized solution $u(t,x)$ of the problem (11),(2) exists. It is continuous in Π_T (for $t>0$).

Class of Correctness

Definition 2. The generalized solution of the problem (11), (2) belongs to the class of the correctness if

$$\operatorname*{ess\,sup}_{t\in[0,T]} \left[\int_0^{+\infty} x^{\frac{3}{2}}u^2(t,x)\,dx + \int_{-\infty}^{+\infty} u^2(t,x)\,dx\right] = M[u,T] < \infty.$$

Theorem 2(continuous dependence on initial data). Let $u(t,x)$ and $v(t,x)$ be generalized solutions of the problem (11),(2) in the sense of the def.2 with initial functions $u_0(x)$ and $v_0(x)$ respectively. Denote $\rho(x)=\min[1,e^{x}]$, $x\in R^1$. Then

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{-\infty}^{+\infty} \rho(x)\big(u(t,x)-v(t,x)\big)^2dx \le$$
$$\le c\big(T,M[u,T],M[v,T]\big)\int_{-\infty}^{+\infty}\big(u_0(x)-v_0(x)\big)^2dx. \qquad (14)$$

Consequence. The generalized solution of the problem (11), (2) in the sense of the def.2 is unique.

Theorem 3(existence in the class of correctness). Assume that

$$\int_0^{+\infty} x^{\frac{3}{2}}u_0^2(x)\,dx + \int_{-\infty}^{+\infty} u_0^2(x)\,dx = M_0 < \infty. \qquad (15)$$

Then the generalized solution $u(t,x)$ of the problem (11),(2) in the sense of def.2 exists. It is continuous for $t>0$ and for $\forall a\in R^1$

$$\sup_{x\geq a} |u(t,x)| \le c(a,T,M_0)\,t^{-\frac{1}{3}},$$
$$\int_a^{+\infty} |u(t,x)-u_0(x)|\,dx \to 0 \text{ as } t\to+0.$$

Theorem 4(regularity). Let the assumption (15) of theorem 3 be fulfilled and for some $\beta>0$ and integer numbers $p\geq 0$, $q\geq 0$

$$\int_0^{+\infty} x^{3p+q+\frac{1}{2}+\beta}u_0^2(x)\,dx < \infty.$$

Then the generalized solution $u(t,x)$ of the problem (11),(2) in the sense of def.2 has in Π_T (for $t>0$) continuous derivatives of the form $\partial^{k+\ell} u(t,x)/\partial t^k \partial x^\ell$, $0 \le k \le p$, $0 \le \ell \le q$.

In connection with the theorem 2 note that the Cauchy problem for the Burgers's equation (4) (dissipation dominates over dispersion) has the unique solution $u^\varepsilon(t,x)$ in the classes without any restrictions on the growth as $|x| \to \infty$. To understand this it is sufficient to remark that the function $w^\varepsilon(t,x) = exp[-U^\varepsilon/2\varepsilon]$, where

$$U^\varepsilon(t,x) = \int_{(0,0)}^{(t,x)} [\varepsilon u^\varepsilon_x - \tfrac{1}{2}(u^\varepsilon)^2]\, dt + u^\varepsilon dx,$$

satisfies the heat equations $w^\varepsilon_t = \varepsilon w^\varepsilon_{xx}$; as a consequence of the D.V.Widder's result [16], the positive function $w^\varepsilon(t,x)$ (as well as $u^\varepsilon(t,x)$) is unique defined by its initial data.

Analogous results are proved in the case of the first boundary value problem for the equation (11) in $\Pi^+_T = (0,T) \times (0,+\infty)$.

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