Josef Král

Potential flows

In: Michal Greguš (ed.): Equadiff 5, Proceedings of the Fifth Czechoslovak Conference on Differential Equations and Their Applications held in Bratislava, August 24-28, 1981. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982. Teubner-Texte zur Mathematik, Bd. 47. pp. 198--204.

Persistent URL: http://dml.cz/dmlcz/702288

POTENTIAL FLOWS

Josef Král
Praha, Czechoslovakia

The role played by the Gauss bell-shaped function $(2\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2t}}$ of the variable $x \in R^n$ in connection with the heat equation is well known; for any Borel set $E \subset R^n$ the integral

$$(1) \qquad \int_E (2\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2t}} dx$$

may be interpreted as the quantity of heat situated in E at the instant $t > 0$ in consequence of the action of a point source at the origin which emanated the unit quantity of heat at the instant 0. Impulses coming from various investigations in mathematics and physics led to extensions of the Gauss measure defined by (1) on Borel sets $E \subset R^n$ to infinite dimensional spaces (cf. [9], [18], [21], [22], [25]). Following L. Gross [11] - [13] we first briefly recall some considerations connected with the concept of the so-called Wiener space. Let H be a real separable Hilbert space with the scalar product $(.,.)$. A cylinder set in H is a set of the form $C = \{h \in H; [(h,h_1),\ldots,(h,h_n)] \in E\}$ where $h_1,\ldots,h_n$ is a finite orthonormal system in H and E is a Borel set in R^n. For such a C the quantity (1) (which actually depends on C only and not on the choice of the orthonormal system describing it) will be denoted by $\mu_t(C)$. The set function μ_t is additive on the algebra of cylinder sets but it fails to be countably additive if $\dim H = \infty$. This indicates the idea of completing H with respect to a new norm $||\cdots||$. A norm $||\ldots||$ on H is termed measurable if for each $\varepsilon > 0$ there is a finite dimensional projection $P_\varepsilon : H \longrightarrow H$ such that $\mu_1(C_P) < \varepsilon$ for $C_P = \{h \in H; ||Ph|| > \varepsilon\}$ whenever P is a finite dimensional projection orthogonal to P_ε. If B denotes the completion of H with respect to a fixed measurable norm $||\ldots||$ then the natural inclusion $H \subset B$ is continuous and the pair (H,B) is called an abstract Wiener space. If B^* is dual to B and $\langle .,. \rangle$ is the pairing between B^* and B then any $y^* \in B^*$ represents, by restriction, a continuous linear functional on H and as such may be identified, via the Riesz representation theorem, with the corresponding element $y \in H$ determined by the equality $\langle y^*,h \rangle = (y,h)$, $h \in H$. We thus arrive at the (dense) inclusions $B^* \subset H \subset B$. Given a Borel set $E \subset$

$\subset R^n$ and a linearly independent system $\{y_1,\dots,y_n\} \subset B^*$ we may form the cylinder $C = \{x \in B;\ [\langle y_1,x\rangle,\dots,\langle y_n,x\rangle] \in E\}$ whose trace on H is $C_H = \{h \in H;\ [(y_1,h),\dots,(y_n,h)] \in E\}$. (Before calculating $\mu_t(C_H)$, one should note that $\{y_1,\dots,y_n\}$ need not be orthonormal in H.) Defining $p_t(C) = \mu_t(C_H)$ we obtain a set function p_t on the algebra of all cylinders C in B; now p_t possesses a unique countably additive extension to the σ-algebra of all Borel sets in B. The Wiener space (H,B) equipped with the probability measures p_t $(t > 0)$ represents a natural playground for many potential theoretic investigations. The measures $\{p_t\}_{t>0}$ form a convolution semigroup which permits to solve suitably formulated Cauchy problem where the initial conditions are given by functions (cf. [12], [20]) or by measures (compare [23]). They also generate the fundamental solution of the heat conduction operator $\Omega = \partial_t - \frac{1}{2}\Delta$ in the space-time $A = R^1 \times B$ whose elements will be written in the form $a = [t,b]$ with $t \in R^1$, $b \in B$ and normed by $(|t|^2+||b||^2)^{1/2}$; H will be identified with the subspace $\{0\} \times H$ in A. Let us define γ on Borel sets $M \subset A$ by $\gamma(M) = \int_0^\infty p_t(\{b \in B;\ [t,b] \in M\})dt$. Then $\gamma \geq 0$ is a measure which is finite on sets that are upper bounded in time. If $\tilde{\Omega}$ denotes the formal adjoint to Ω then $\int_A \tilde{\Omega}\phi d\gamma = \phi(0)$ for all test functions ϕ to be defined below; this means that $\Omega\gamma = \delta$ (= the unit Dirac measure concentrated at the origin of A). Let us recall that a function $f : A \longrightarrow R^1$ defined in some neighborhood of $a \in A$ is termed quasidifferentiable at a if there is a $Df(a) \in A^*$ such that, for each continuous $\alpha : (-1,1) \longrightarrow A$ differentiable at 0 with $\alpha(0) = a$, the composition $t \longmapsto f(\alpha(t))$ has a derivative at 0 given by $\langle Df(a),\alpha'(0)\rangle$. We shall deal with uniformly continuous functions ϕ with bounded support that are everywhere quasidifferentiable with $D\phi$ bounded (in the dual norm) and $[a_1,a_2] \longmapsto \langle D\phi(a_1),a_2\rangle$ continuous on $A \times A$; such a ϕ will be called test function if, in addition, for any fixed a the function $h \longmapsto \phi(a+h)$ is twice Fréchet differentiable at the origin of H, the corresponding second order differential $D^2\phi(a)$ (considered as a linear operator on H) is of trace class and the map $a \longmapsto D^2\phi(a)$ into the space of all trace class operators on H is bounded and continuous. For any test function ϕ the laplacian $\Delta\phi(a) = \mathrm{Trace}\, D^2\phi(a)$ as well as $\partial_t\phi(a) = \langle D\phi(a),[1,0]\rangle$, $\langle\hat{D}\phi(a),b\rangle = \langle D\phi(a),[0,b]\rangle$ $(b \in B)$ are defined and $\tilde{\Omega}\phi = -(\partial_t + \frac{1}{2}\Delta)\phi$ is a bounded continuous function with bounded support. It is an important consequence of results of [12], [8], [10] that the class of all test functions is sufficiently rich to admit construction of partitions of unity subordinated to open covers of the space; any uniformly continuous function with bounded support is a uniform limit of a sequence of test functions. Convolutions of γ with other measures generate potentials useful for treating various boundary value problems. As an illustrative example we shall consider the so-called Fourier problem (cf. [15], [19] for related finite dimensional considerations). Let us fix a $T > 0$ and an open set $\Omega \subset B$ with boundary $\partial\Omega$. Put $V = (0,T) \times \Omega$, $Q = \langle 0,T) \times \partial\Omega$ and denote by $\mathcal{B}'$ the space of all signed Borel measures ν on A whose variation $|\nu|$ fulfils $|\nu|(A \setminus Q) = 0$. With each $\nu \in \mathcal{B}'$ we associate its potential $u \equiv u_\nu = \nu * \gamma$ defined on the σ-algebra $\mathcal{M}_T$ of all Borel sets $M \subset (-\infty,T\rangle \times B$ by $u(M) = \int \nu(M-a)\gamma(da)$. It follows from basic properties of γ that u is H-differentiable: there exists a countably additive H-valued measure $M \longrightarrow \operatorname{grad} u(M) \in H$ on $\mathcal{M}_T$ such that, for any $h \in H$ and $M \in \mathcal{M}_T$, $\lim_{t \to 0+} t^{-1}[u(M+th)-u(M)] = (h, \operatorname{grad} u(M))$ (cf. [1] for differentiability of measures). If $\phi \in \mathcal{D}_T$ (= the class of all test functions with support in $(-\infty,T) \times B$) then $\hat{D}\phi : A \longrightarrow B^* \subset H$ can be integrated against the vector-valued measure $\operatorname{grad} u : \mathcal{M}_T \longrightarrow H$ and the bilinear integral (cf. [2]) extended over V permits to define the functional $\mathbb{H}\nu$ over $\mathcal{D}_T$ by

$$\langle \phi, \mathbb{H}\nu \rangle = \int_V (\hat{D}\phi, \operatorname{grad} u_\nu) - 2\int_V \partial_t \phi(a) u_\nu(da) , \quad \phi \in \mathcal{D}_T .$$

$\mathbb{H}\nu$ will be called the heat flow associated with ν. It is easily seen that $\langle \phi, \mathbb{H}\nu \rangle = 0$ if $\phi \in \mathcal{D}_T$ and $Q \cap \operatorname{spt} \phi = \emptyset$. The physical significance of $\mathbb{H}\nu$ is best illustrated in finite dimensional case when $\operatorname{grad} u_\nu$ has a density ∇u_ν (with respect to the Lebesgue measure) extending from V to a continuous function on $V \cup Q$; if $\partial\Omega$ is smooth, N is the unit exterior normal to V and σ is the surface measure, then

$$\langle \phi, \mathbb{H}\nu \rangle = \int_Q \phi (N \cdot \nabla u_\nu) d\sigma , \quad \phi \in \mathcal{D}_T .$$

In general case $\mathbb{H}\nu$ need not be representable by a (signed) measure μ in the sense that $\langle \phi, \mathbb{H}\nu \rangle = \int \phi d\mu$, $\phi \in \mathcal{D}_T$. If there is such a μ then it is uniquely determined by the requirement $|\mu|(\langle T,+\infty) \times B) = 0$; under this condition it belongs to $\mathcal{B}'$ and, as usual, will be identified with $\mathbb{H}\nu$. In order to get geometric conditions on Ω guaranteeing $\mathbb{H}\nu \in \mathcal{B}'$ for every $\nu \in \mathcal{B}'$ it is useful to adopt the following definition (compare [14], [17]).

DEFINITION. *If* C *is a Banach space,* $M \subset C$ *is a Borel set and* $\omega : I \to C$ *is a simple differentiable path-curve defined on an interval* $I \subset R^1$ *, then* $c \in C$ *is termed a hit of* ω *on* M *if there is a* $t_c \in I$ *with* $\omega(t_c) = c$ *and for each neighborhood* $\mathcal{U}$ *of* c *in* C *both* $\{t \in I;\ \omega(t) \in \mathcal{U} \cap M\}$ *and* $\{t \in I;\ \omega(t) \in \mathcal{U} \setminus M\}$ *have positive linear measure.*

This definition enables us to formulate the required condition in terms of the measure p_1 as follows.

THEOREM 1. *Given* $R > 0$ *,* $x \in \partial\Omega$ *and* $b \in B$ *we denote by* $n_R(x,b)$ *the total number of all hits of* $\{x+tb;\ 0 < t < R\}$ *on* $\Omega \subset B$ *. Then the function* $b \mapsto n_R(x,b)$ *is measurable and*

$$\sup_{x \in \partial\Omega} \int_B n_{\sqrt{T}}(x,b)p_1(db) < \infty \tag{2}$$

is a necessary and sufficient condition guaranteeing that $\mathbb{H}\nu \in \mathcal{B}'$ *for each* $\nu \in \mathcal{B}'$ *. If* (2) *holds then* $\mathbb{H} : \nu \mapsto \mathbb{H}\nu$ *is a bounded operator on the space* $\mathcal{B}'$ *equipped with the norm* $||\nu|| = |\nu|(Q)$ *.*

The norm of the operator $\mathbb{H}$ can also be simply evaluated. If $x \in \partial\Omega$ and $R > 0$ then $\int_B n_R(x,b)p_1(db) < \infty$ implies measurability of the set Ω_x formed by those $b \in B$ for which there is a $\delta \equiv \delta(x,b) > 0$ with $x+tb \in \Omega$ for a.e. $t \in (0,\delta)$. Assuming (2) and writing $d_\Omega(x) = p_1(\Omega_x)$ (which in finite dimensional case reduces to the Lebesgue density of Ω at x) we have

$$||\mathbb{H}|| = 2 \sup_{x \in \partial\Omega} \{d_\Omega(x) + \int_B n_{\sqrt{T}}(x,b)p_1(db)\} .$$

In particular, $||\mathbb{H}|| \leq 2$ for convex Ω .

The condition (2) can be somewhat simplified if we put $\Gamma = \{\theta \in B;\ ||\theta|| = 1\}$ and denote by σ_1 the probability measure on Γ which is the image of p_1 under the projection $b \mapsto \frac{b}{\|b\|}$ of $B \setminus \{0\}$ onto Γ . Then

$$\infty > \sup_{x \in \partial\Omega} \int_\Gamma n_\infty(x,\theta)\sigma_1(d\theta) \quad \left(= \sup_{x \in \partial\Omega} \int_B n_\infty(x,b)p_1(db)\right)$$

is sufficient (and also necessary in finite dimensional case) for validity of (2).

In what follows we always assume (2). The Fourier problem has then the following natural weak formulation: Given $\mu \in \mathcal{B}'$, determine a $\nu \in \mathcal{B}'$ with $\mathbb{H}\nu = \mu$. General description of the range of $\mathbb{H}$ is not known. However, sufficient geometric conditions on Ω guaranteeing invertibility of $\mathbb{H}$ can be established. If $P \subset \partial\Omega$ is a Borel set and $\mathcal{B}'_P$ denotes the subspace of all $\nu \in \mathcal{B}'$ with $|\nu|(\langle 0,T) \times P) = |\nu|(Q)$, then the following result holds.

<u>THEOREM 2.</u> *If*

$$(3) \qquad \lim_{r\to 0+} \sup_{x\in P} \{|2d_\Omega(x)-1| + 2\int_B n_r(x,b)p_1(db)\} < 1 ,$$

then for every $\nu \in \mathcal{B}'_P$ *there exists a unique* $\mu \in \mathcal{B}'_P$ *such that* $\mathbb{H}\mu(M) = \nu(M)$ *for each Borel* $M \subset \langle 0,T) \times P$.

If all the functions $n_\infty(x,.)$ with $x \in P$ are dominated by a common p_1-integrable function (or if $\dim H < \infty$) then the integral occurring in (3) may equivalently be replaced by $\int_\Gamma n_r(x,\theta)\sigma_1(d\theta)$. Of course, main interest consists in $P = \partial\Omega$ when (3) implies the existence of a bounded inverse to $\mathbb{H}$ on $\mathcal{B}'$. As pointed out by W. Wendland, (3) then appears to be rather restrictive for certain non-convex domains even in the finite dimensional case. It is therefore useful to replace $n_r(x,b)$ by a more general quantity counting hits with a suitable weight. Let $\Psi \geq 0$ be a lower semicontinuous function on $\partial\Omega$. Given x, $b \in B$ and $R > 0$ we put $n_R^\Psi(x,b) = \sum_{0<t<R} \Psi(x+tb)$, where the sum extends over those t for which $x+tb$ is a hit of $\{x+tb;\ 0 < t < R\}$ on $\Omega \subset B$. If Ψ is bounded and bounded away from zero, then (2) is equivalent to

$$\sup_{x\in\partial\Omega} \int_B n^\Psi_{\sqrt{T}}(x,b)p_1(db) < \infty ;$$

on the other hand, the condition

$$\lim_{r\to 0+} \sup_{x\in\partial\Omega} \{|2d_\Omega(x)-1| + 2\Psi^{-1}(x)\int_B n_r^\Psi(x,b)p_1(db)\} < 1 ,$$

which guarantees the existence of a bounded inverse to $\mathbb{H}$ on $\mathcal{B}'$, can be shown to be more general than (3) for $P = \partial\Omega$. Restrictions imposed on Ψ may be relaxed and generalizations of the heat flow operator $\mathbb{H}$ can be introduced acting on broader spaces of measures which need not have bounded variation.

Suitable extensions of the quantity $n_r(x,b)$ find application in treating boundary value problems for time-variable domains. If $\hat{\mathcal{V}} \subset (0,T) \times B$ is a general open set and $\hat{\mathcal{Q}} = \partial\hat{\mathcal{V}} \cap \{(0,T) \times B\}$, then for any $z = [u,x] \in \hat{\mathcal{Q}}$ the quantity $n(z,b)$ counting all the hits of the parabola $\{[u+t^2, x+tb];\ t > 0\}$ on $\hat{\vartheta} \subset A$ is a measurable function of the variable $b \in B$. The condition

$$(4) \qquad \sup_{z \in \hat{\vartheta}} \int_B n(z,b) p_1(db) < \infty$$

appears to be an adequate extension of (2) for this type of domains. This was shown in general investigations on integral representability of solutions of boundary value problems for the heat equation and the adjoint heat equation on time variable domains in finite dimensional spaces due to M. Dont and J. Veselý (cf. [5] - [7], [24]).

As shown by L. Gross [12] (cf. also [4]), Newtonian potentials on B can naturally be defined with help of the measure which is the image of γ under the projection $[t,b] \to b$ of A onto B. In finite dimensional case results related to those described above were obtained for Newtonian potentials in [3], [14] (cf. also [16] for comments and references).

REFERENCES

[1] В. И. Авербух, О. Г. Смолянов, С. В. Фомин: Обобщенные функции и дифференциальные уравнения в линейных пространствах, Труды Московского мат. об-ва 24 (1971), 133-174.

[2] R. G. Bartle: A general bilinear vector integral, Studia Mathematica XV (1955), 337-352.

[3] Ю. Д. Бураго, В. Г. Мазья: Некоторые вопросы теории потенциала и теории функций для областей с нерегулярными границами, Записки научных семинаров ЛОМИ т. 3, 1967.

[4] R. Carmona: Infinite dimensional Newtonian potentials, Proc. Conference "Probability theory on vector spaces II", Lecture Notes in Math. vol. 828, Springer-Verlag 1980, 30-43.

[5] M. Dont: On a heat potential, Czechoslovak Math. J. 25 (1975), 84-109.

[6] M. Dont: On a boundary value problem for the heat equation, Czechoslovak Math. J. 25 (1975), 110-133.

[7] M. Dont: Flows of heat and time moving boundary, to appear.

[8] X. Fernique: Intégrabilité des vecteurs Gaussiens, C.R.Acad. Sci. Paris 270 (1970), sér. A, 1698-1699.

[9] И. М. Гельфанд, Н. Я. Виленкин: Обобщенные функции вып. 4, Москва 1961.

[10] V. Goodman: Quasi-differentiable functions on Banach spaces, Proc. Amer. Math. Soc. 30 (1971), 367-370.

[11] L. Gross: Classical analysis on a Hilbert space, Proc. Conf. "Analysis in Function Space", M.I.T. Press 1964, 51-68.

[12] L. Gross: Potential theory on Hilbert space, Journal of Functional Analysis 1 (1967), 123-181.

[13] L. Gross: Abstract Wiener spaces, Proc. Fifth Berkeley Symp. on Math. Stat. and Prob. vol. II, Univ. of California Press 1967, 31-42.

[14] J. Král: The Fredholm method in potential theory, Trans. Amer. Math. Soc. 125 (1966), 511-547.

[15] J. Král: Flows of heat and the Fourier problem, Czechoslovak Math. J. 20 (1970), 556-598.

[16] J. Král: Integral operators in potential theory, Lecture Notes in Math. vol. 823, Springer-Verlag 1980.

[17] J. Král: Boundary behavior of potentials, Proc. Conf. Equadiff IV, Lecture Notes in Math. vol. 703, Springer-Verlag 1979, 205-212.

[18] Hui-Hsiung Kuo: Gaussian measures in Banach spaces, Lecture Notes in Math. vol. 463, Springer-Verlag 1975.

[19] I. Netuka: A mixed boundary value problem for heat potentials, Comment. Math. Univ. Carolinae 19 (1978), 207-211.

[20] M. Ann Piech: A fundamental solution of the parabolic equation on Hilbert space, Journal of Functional Analysis 3 (1969), 85-114.

[21] А. В. Скороход: Интегрирование в гильбертовом пространстве, Москва 1975.

[22] Г. Е. Шилов, Фан Дык Тинь: Интеграл, мера и производная на линейных пространствах, Москва 1967.

[23] А. В. Угланов: Уравнение теплопроводности для мер в оснащенном гильбертовом пространстве. Вестник Московского ун-та 1971, Н° -1, 52-60.

[24] J. Veselý: On a generalized heat potential, Czechoslovak Math. J. 25 (1975), 404-423.

[25] Xia Dao-Xing: Measure and integration theory on infinite-dimensional spaces, Academic Press, New York and London 1972.