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ON A DEGENERATE PARABOLIC BOUNDARY VALUE PROBLEM

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A nonlinear degenerate parabolic boundary value problem is considered in the form

$$\text{(E)} \qquad \alpha(x)\frac{\partial u}{\partial t} - \sum_{i=1}^{N}\frac{\partial}{\partial x_j} a_i(x,u,\nabla u) + a_0(x,u,\nabla u) = f(x,t)$$

on $\Omega \times (0, T) \geq 0$ where $\Omega \subset R^N$ is a bounded domain and $\alpha(x) \geq 0$ is a measurable function on Ω. A corresponding Dirichlet boundary condition and initial condition $u(x,0) = u_0(x)$ is assumed.

Together with (E) a corresponding parabolic variational inequality is considered. The problems of the existence uniqueness of the solution in the corresponding functional spaces is solved. Two cases are considered:

I. $\alpha(x) > 0$ for a.e. $x \in \Omega$

II. $\alpha(x) = 0$ in $\Omega_2 \subset \Omega$ (Ω_2 is an open subset in Ω) and $\alpha(x) > 0$ a.e. in $\Omega_1 = \bar{\Omega} - \Omega_2$ where the boundaries $\partial\Omega$, $\partial\Omega_1$, $\partial\Omega_2$ are Lipschitz continuous.

The problem (E) and the corresponding variational inequality we set in an abstract form.

Case I. Let X be reflexive B - space with its dual X^* with the corresponding norms $\|\cdot\|_X$, $\|\cdot\|_{X^*}$. The duality between X^* and X we denote by $\langle \cdot , \cdot \rangle$.
Let H_1, H_2 be the real Hilbert spaces with the corresponding norms $\|\cdot\|_1$, $\|\cdot\|_2$. Suppose that $[\cdot , \cdot]$ is a continuous bilinear form between the elements of H_1 and H_2 satisfying

$$|[u, v]| \leq \|u\|_2 \, \|v\|_1 \qquad \text{for } u \in H_2, \ v \in H_1 .$$

We identify H_1, H_2 with their duals. A linear operator $G \in L(H_1, H_2)$ is considered. Let $A : X \to X^*$ be a monotone operator. We assume that $X \cap H_1$ is a nonempty B - space with the standard norm $\|\cdot\|_{V \cap H_1} = \|\cdot\|_X + \|\cdot\|_1$. Moreover we assume that $\|u\|_1 = 0$

implies $\| u \|_X = 0$ for $u \in V \cap H_1$. By I we denote the interval $< 0, T >$, $T < \infty$.

Problem P_1. Let $u_0 \in X \cap H_1$ and $f \in C(I, H_2)$.

Find a $u \in L_\infty(I, X \cap H_1) \cap C(I, H_1)$ such that $u(0) = u_0$

$\frac{du}{dt} \in L_\infty(I, H_1)$ and the identity

$$[G \frac{du(t)}{dt}, v] + < A u(t), v > = [f(t), v] \tag{1}$$

holds for every $v \in X \cap H_1$ and a.e. $t \in I$.

Problem P'_1. Let K be a closed convex subset in $X \cap H_1$, $u_0 \in K$ and $f \in C(I, H_2)$. Find a $u \in L_\infty(I, K) \cap C(I, H_1)$ such that $u(0) = u_0$, $\frac{du}{dt} \in L_\infty(I, H_1)$ and the inequality

$$[G \frac{du(t)}{dt}, v - u(t)] + < A u(t), v - u(t)> \geq [f(t), v-u(t)] \tag{1'}$$

holds for all $v \in K$ and a.e. $t \in I$.

The identity (1) can be interpreted as the corresponding operator equation in $X^* + H_2$. When $X \cap H_1$ is a dense set in X and H_1 then the corresponding operator equation can be interpreted in H_2.

Remark. As an application for the problem (E) we set $H_1 = L_2(\Omega, \alpha)$, $H_2 = L_2(\Omega, \alpha^{-1})$ (weighted spaces with respect to α, α^{-1}, respectively). We put $[u, v] = \int_\Omega u\, v\, dx$, $G u = \alpha(x) u$ and $A : X \to X^*$ we define by the form

$$< A u, v > = \int_\Omega \{ \sum_{i=1}^{N} \frac{\partial v}{\partial x_i} a_i (x, u, \nabla u) + v\, a_0(x, u, \nabla u) \} dx$$

for $u, v \in X \equiv \overset{\circ}{W}{}^1_p (\Omega)$

under the assumption $|a_i (x, \xi)| \leq C(1 + |\xi|^{p-1})$ $(p>1)$

for $i = 0, 1, \ldots, N$. In general case H_1 is not necessarily separable and $X \cap H_1$ is not dense set in X and H_1.

When $\int_\Omega \alpha(x) |x_1|^{i_1} \ldots |x_N|^{i_N} dx < \infty$ for all $0 \leq i_\ell < \infty$, $\ell = 1, \ldots, N$, then the space H_1 is separable and $X \cap H_1$ is a dense set in X and H_1.

The problems P_1 and P_1' we solve under the following assumtions:

(2) $A : X \to X^*$ is bounded and demicontinuous;

(3) $\langle Au - Av, u - v \rangle \geq 0$ for $u, v \in X$;

(4) $f \in C(I, H_2)$ with $\operatorname{Var}_I (f, H_2) < \infty$,

where $\operatorname{Var}_I (f,H_2) = \sup_{\{t_i\}_0^m} \sum_{i=1}^{m} \| f(t_i) - f(t_{i-1}) \|_2$ and $\{t_i\}_0^m$ is a finite division of I.

(5) $(\langle Au, u \rangle + \alpha \| u \|_1^2) / \| u \|_X \to \infty$ for $\| u \|_X \to \infty$,

for suitable $\alpha \geq 0$.

In the case of the problem P_1' we replace (5) by the assumption

(5′) $\exists\, v_0 \in K : \langle A u, u - v_0 \rangle / [u]_X \to \infty$ for $[u] \to \infty$

where $[u]$ is a seminorm in X satisfying : $\exists\, \beta > 0$ such that $[u]_X + \beta \| u \|_1 \geq C \| u \|_X$ for $u \in X \cap H_1$.

We assume

(6) $[G u, v] \leq \| u \|_1 \| v \|_1$ and $[G u, u] = \| u \|_1^2$.

<u>Theorem 1.</u>

i/ Let $u_0 \in X \; H_1$ and let (2) - (6) be satisfied. If

(7) $\sup_{\| v \|_1 \leq 1,\ v \in X \cap H_1} | \langle A u_0, v \rangle | < \infty$

holds then there exists the unique solution of the problem P_1.

ii/ Let $u_0 \in K$ and let (2) - (7) be satisfied.

Then there exists the unique solution of the problem P_1'.

The method of the proof is based on Rothés method /method of lines/. Let $u_i (i=1, \ldots, n)$ be corresponding problems

(8) $[G u, v] + h \langle Au, v \rangle = h [f(t_i), v] - [G u_{i-1}, v]$

(8′) $([G u, v - u] + h \langle Au, v-u \rangle \geq h [f(t_i), u-v] - [G u_{i-1}, v-u])$

where $h = \frac{T}{n}$, $t_j = jh$ $(j = 1, \ldots, n)$. By means of u_i $(i=1, \ldots, n)$ we construct the function

$$u_n(t) = u_{j-1} + h^{-1}(t-t_{j-1})(u_j - u_{j-1}), \quad t_{j-1} \le t \le t_j,$$
$$j = 1, \ldots, n.$$

On the base of (8) ((8')) using (3) - (7) we obtain (similarly as in [2-4]) the apriori estimates

$$\left\| \frac{du_n(t)}{dt} \right\|_1 \le C, \quad \| u_n(t) \|_{X \cap H_1} \le C \quad \text{(C is independent on t and n)}$$

which allows us to take limit for $n \to \infty$ in the approximate identity

$$(9) \qquad \left[G \frac{du_n(t)}{dt}, v \right] + \langle A\bar{u}_n(t), v \rangle = \langle f_n(t), v \rangle$$

which we obtain trom (7) where $\bar{u}_n(t) = u_j$ for $t_{j-1} < t \le t_j$, $\bar{u}_n(0) = u_0$ is the step function. Analogously we construct $\bar{f}_n(t)$. Similarly we proceed in (8').

Theorem 2.

Let (2-7) be satisfied and $f: I \to H_2$ is Lipschitz continuous, i.e., $\| f(t) - f(t') \| \le C |t - t'|$.

Then the estimate holds

$$\| u_n(t) - u(t) \|^2_{C(I, H_1)} \le \frac{C}{n}$$

where $u(t)$ is the solution of the problem $P_1(P_1')$.

The case II. To give an abstract formulation corresponding to this case we follow the concept of [1].
Let A, G, H_1, H_2 and X be as in the case I. In the case II. they correspond to the subset Ω_1. Let Y be a reflexive space with its dual Y^* and duality $\langle ., . \rangle_*$. We consider a demicontinuous, coercive and strongly monotone operator $B : Y \to Y^*$ satisfying

$$(10) \qquad \langle Bu - Bv, u - v \rangle_* \ge C \| u - v \|^p_Y \qquad (p > 1)$$

$$(11) \qquad \langle By, y \rangle_* / \| y \|_Y \to \infty \quad \text{for} \quad \| y \|_Y \to \infty$$

(Y and B correspond to the subset Ω_2). We define Cartesian product $W = X \cap H_1 \times Y$ with the standard norm. Let $T : W \to W^*$ be the operator defined by the form

$$(Tu, v) = \langle Au_1, v_1 \rangle + \langle Bu_2, v_2 \rangle_*$$
$$\text{for} \quad u = \{u_1, u_2\}, \; v = \{v_1, v_2\} \in W.$$

We denote by $(f,v) = [f_1, v_1] + \langle f_2, v_2 \rangle_*$ for $f_1 \in H_2$, $f_2 \in Y^*$.
Let V be a (suitable) nonempty subspace of W.

Problem P_2. Let $f_1 \in C(I, H_2)$, $f_2 \in C(I, Y^*)$, $u_0 \in X \cap H_1$. To look for $u \in L_\infty(I, V)$ $(u(t) = \{u_1(t), u_2(t)\})$ such that

$$\frac{du_1}{dt} \in L_\infty(I, H_1),\ u_1(0) = u_0 \text{ and the identity}$$

$$\left[G \frac{du_1(t)}{dt}, v_1\right] + (T\,u(t), v) = (f(t), v) \quad \text{for all } v \in V.$$

Analogously (as in the case I) we define problem P_2' corresponding to the variational inequality.

Example. Considering problem (E) under the growth assumption $|a_i(x, \xi)| \le C(1 + |\xi|^{p-1})$ $(i = 0,1,\dots,N;\ p > 1)$ we set:

$$H_1 = L_2(\Omega_1, \alpha), \qquad H_2 = L_2(\Omega_1, \alpha^{-1}),$$

$$X = \{v \in W_p^1(\Omega_1) : v=0 \text{ on } \partial\Omega \cap \partial\Omega_1\}$$

$$Y = \{v \in W_p^1(\Omega_2) : v=0 \text{ on } \partial\Omega \cap \partial\Omega_2\}$$

$$\langle Au_1, v_1 \rangle = \int_{\Omega_1} \left\{ \sum_{i=1}^N a_i(x, u_1, \nabla u_1) \frac{\partial v_1}{\partial x_i} + a_0(x, u_1, \nabla u_1) v_1 \right\} dx$$

$$\langle Bu_2, v_2 \rangle_* = \int_{\Omega_2} \left\{ \sum_{i=1}^N a_i(x, u_2, \nabla u_2) \frac{\partial v_2}{\partial x_i} + a_0(x, u_2, \nabla u_2) v_2 \right\} dx$$

The elements $u = \{u_1, u_2\} \in W$ we represent as a function on Ω such that $u = u_1$ on Ω_1 and $u = u_2$ on Ω_2. We define $V \equiv \overset{\circ}{W}{}_p^1(\Omega) \cap L_2(\Omega_1, \alpha)$.

Theorem 3 .

Let (2) - (7), (10), (11) be satisfied. If $\operatorname{Var}_I (f_1, H_2) < \infty$, $\| f_2(t) - f_2(t') \|_{Y^*} \le C\,|t - t'|$ $(t, t' \in I)$ holds then there exists the unique solution of the problem P_2.

Similar result can be obtained for the problem P_2' corresponding to the variational inequality.

R E F E R E N C E S

[1] Zlámal, M.: Finite element solution of quasistationary nonlinear magnetic field, to appear.

[2] Kačur, J.: Application of Rothés method to nonlinear evolution equations. Mat.Čas. 25, 1975, 63-81.

[3] Kačur, J.: Method of Rothe and nonlinear parabolic boundary value problems of arbitrary order. Czech. Mat.J., V.28 (103), 1978, 507 - 524.

[4] Kačur, J.; Bock, I.: Application of Rothés method to parabolic variational inequalities. To appear.