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APPROXIMATION OF CONTACT PROBLEMS WITH FRICTION

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1. Introduction

A great number of technical problems lead to the study of the behaviour of complicated structures, composed of two or more deformable bodies in mutual contact. Contact problems without friction have been studied by many authors from mathematical as well as computational point of view. Here, we describe the approximation of the simplest model, involving friction, namely model "with a given friction", definition of which is given in the next section. For the sake of simplicity we restrict ourselves to the plane case only, when an elastic body is unilateraly supported by a rigid foundation. All these results can be very easy extended to the case of two (or more) elastic bodies in contact. A detailed analysis of all results, presented below, can be found in [2]. The approximation of our problem is based on the so called reciprocal variational formulation. The use of some other variational formulations for the approximation of contact problems with friction is studied in [3, 4].

2. Setting of the problem

Let an elastic body be represented by a domain $\Omega \subset R^2$, the Lipschitz boundary of which is decomposed into two non-empty parts Γ_u and Γ_K. A displacement field $u = (u_1, u_2)$ is said to be a classical solution of the contact problem with a given friction, if

$$(2.1)\quad \frac{\partial \tau_{ij}(u)}{\partial x_j} + F_i = 0, \quad i = 1,2 \quad \text{in } \Omega,$$

i.e. u is in the equilibrium state with body forces $F = (F_1, F_2)$ and it satisfies:

homogeneous boundary conditions

$$(2.2)\quad u_i = 0, \quad i = 1,2 \quad \text{on } \Gamma_u,$$

unilateral conditions

$$(2.3)\quad u_n = u\cdot n \leq 0,\ T_n(u) \equiv \tau_{ij}(u)n_i n_j \leq 0,\ u_n T_n(u) = 0 \quad \text{on } \Gamma_K,$$

<u>friction conditions</u>

$$(2.4)\quad \begin{cases} |T_t(u)| \leq g,\ T_t(u) = \tau_{ij}(u)n_i t_j, \\ \text{if } |T_t(u)(x)| < g(x) \Longrightarrow u_t(x) = u_i t_i = 0, \\ \text{if } |T_t(u)(x)| = g(x)\ \exists\, \lambda \geq 0,\ T_t(u)(x) = -\lambda u(x),\ x \in \Gamma_K. \end{cases}$$

τ_{ij} are components of the stress tensor τ, related to the strain tensor ε by means of linear Hooke's law, n and t denote the unit normal and tangential vector to $\partial\Omega$.

In order to give the variational formulation of the problem in question, we introduce a Hilbert space

$$V = \{ v = (v_1, v_2) \in (H^1(\Omega))^2 \mid \quad v_i = 0,\ i = 1,2,\ \text{ on } \Gamma_u \}$$

and its closed convex subset

$$K = \{ v \in V \mid v_n \leq 0 \text{ on } \Gamma_K \}.$$

Finally denote by $\mathcal{J}$ the functional of total potential energy, given by

$$\mathcal{J}(v) = (\tau_{ij}(v), \varepsilon_{ij}(v))_0 + \int_{\Gamma_K} g|v_t|\,ds - (F_i, v_i)_0,$$

where $(\ ,\ )_0$ denotes L^2-scalar product, $F \in (L^2(\Omega))^2$, $g \in L^\infty(\Gamma_K)$, $g \geq 0$.

<u>Primal variational formulation</u> is defined as the problem of finding a minimiser u of $\mathcal{J}$ over K:

$$(\mathcal{P}_p) \qquad u \in K:\ \mathcal{J}(u) \leq \mathcal{J}(v) \quad \forall v \in K.$$

It is well-known that there exists a unique solution u of $(\mathcal{P}_p)$ (see [1]).

Now, let us introduce the following quadratic functional:

$$\mathcal{S}(\mu_1,\mu_2) = 1/2 \langle \mu_1, G(\mu_1,\mu_2)\cdot n \rangle + 1/2 \langle \mu_2, G(\mu_1,\mu_2)\cdot t \rangle + \\ + \langle \mu_1, G(F)\cdot n \rangle + \langle \mu_2, G(F)\cdot t \rangle ,$$

where $(\mu_1, \mu_2) \in (H^{-1/2}(\Gamma_K))^2$, $\langle\ ,\ \rangle$ denotes the duality pairing between $H^{-1/2}(\Gamma_K)$ and $H^{1/2}(\Gamma_K)$ and $G: V' \to V$ is the Green's operator, associated with the bilinear form $(\tau_{ij}(v), \varepsilon_{ij}(z))_0$ and the space V.

By <u>reciprocal variational formulation</u> we call the problem of finding $\lambda = (\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2$, satisfying

$$(\mathcal{P}_r) \quad \mathcal{S}(\lambda_1, \lambda_2) \leq \mathcal{S}(\mu_1,\mu_2) \quad \forall \quad (\mu_1,\mu_2) \in \Lambda_1 \times \Lambda_2,$$

where

$$\Lambda_1 = H^{-1/2}(\Gamma_K) = \{\mu_1 \in H^{-1/2}(\Gamma_K) \mid \langle \mu_1, v_n \rangle \geq 0 \ \forall \ v \in K\},$$

$$\Lambda_2 = \{\mu_2 \in L^2(\Gamma_K) \mid |\mu_2| \leq g \quad \text{on } \Gamma_K\}.$$

The relation between $(\mathcal{P}_r)$ and $(\mathcal{P}_p)$ is given by

Theorem 2.1. There exists a unique solution of $(\mathcal{P}_r)$. Moreover

$$\lambda_1 = T_n(u), \quad \lambda_2 = T_t(u),$$

where $u \in K$ is the solution of $(\mathcal{P}_p)$.

3. Approximation of the reciprocal variational formulation

Let $\Omega \subset R^2$ be a polygonal domain and $\{\mathcal{T}_h\}$, $h \to 0+$ a regular family of triangulations of $\overline{\Omega}$, which is consistent with the decomposition of $\partial\Omega$ into Γ_u and Γ_K. By V_h we denote the finite-dimensional subspace of V, containing all piecewise linear functions on $\mathcal{T}_h$. Let $\{\mathcal{T}_H\}$ be another family of partitions of Γ_K, nodes of which, denoted by $b_1, \ldots, b_{m(H)}$, don't coincide with boundary nodes of $\mathcal{T}_h$, in general. Λ_{1H} and Λ_{2H} are defined as follows:

$$\Lambda_{1H} = \{\mu_{1H} \in L^2(\Gamma_K), \ \mu^i_{1H} \in P_o(b_i b_{i+1}), \ \mu_{1H} \leq 0 \quad \forall i\},$$

$$\Lambda_{2H} = \{\mu_{2H} \in L^2(\Gamma_K), \ \mu^i_{2H} \in P_o(b_i b_{i+1}), \ |\mu^i_{2H}| \leq g^i \ \forall \ i\},$$

where $\mu^i_{jH} = \mu_{jH}|_{b_i b_{i+1}}$, $j = 1,2$, $P_o(b_i b_{i+1})$ denotes the set of all constant functions on $b_i b_{i+1}$ and g^i is the mean value of g on $b_i b_{i+1}$.

As the explicit form of G, appearing in the definition of $\mathcal{J}$ is not known, in general, G must be approximated. Here, we describe one of the possible approximations. Let A_h be the matrix of rigidity, related to the bilinear form $(\tau_{ij}(v), \varepsilon_{ij}(z))_o$ and to V_h. The approximation G_h of G is now defined as the mapping of V'_h into V_h, represented by the inverse of A_h. So we are led to the following definition:

Definition 3.1. By the approximation of the reciprocal variational formulation of the contact problem with a given friction we mean the problem of finding $(\lambda_{1H}, \lambda_{2H}) \in \Lambda_{1H} \times \Lambda_{2H}$, satisfying

$$(\mathcal{P}_r)_{hH} \quad \mathcal{J}_h(\lambda_{1H}, \lambda_{2H}) \leq \mathcal{J}_h(\mu_{1H}, \mu_{2H}) \ \forall \ (\mu_{1H}, \mu_{2H}) \in \Lambda_{1H} \times \Lambda_{2H},$$

where $\mathcal{J}_h$ is obtained from $\mathcal{J}$ by replacing G by G_h. The analysis of the relation between $(\lambda_{1H}, \lambda_{2H})$ and (λ_1, λ_2) is given in [2].

References

[1] Duvaut, G., Lions, J. L.: Les Inéquations en Mécanique et en Physique, Dunod, Paris 1972.

[2] Haslinger, J., Panagiotopoulos, D.: Reciprocal variational formulation of contact problems with friction and its approximation by finite elements (to appear).

[3] Haslinger, J., Hlaváček, I.: Approximation of the Signorini problem with friction by a mixed finite element method (to appear in JMAA).

[4] Haslinger, J., Tvrdý, M.: Approximation and numerical realization of contact problems with friction (to appear in Apl. Mat.).