ON THE MATHEMATICAL AND NUMERICAL STUDY OF NON-VISCOUS AXIALLY SYMMETRIC CHANNEL FLOWS

Miloslav Feistauer
Prag, Czechoslovakia

The paper is devoted to the solution of steady, non-viscous, generally rotational, incompressible or subsonic compressible, axially symmetric channel flows (detailes and references - see [1, 2]). The study of this problem plays an important role e.g. in the investigation of aerodynamical properties of diffusers of turbines and compressors.

1. Formulation of the flow problem

Let $\Omega \subset R_2$ be a bounded domain lying with its closure $\bar{\Omega}$ in the upper half-plane $x = (x_1, x_2)$, $x_2 > 0$. The boundary $\partial\Omega$ of Ω consists of closed Jordan curves $C_0, \dots, C_k$, $k \geq 0$. By rotating the domain Ω round the axis x_1 we get a three-dimensional domain filled by the fluid. Let $C_i \subset \operatorname{Int} C_0$ for $i = 1, \dots, k$.

The flow problem (we shall denote it (P1)) can be formulated as follows: To find $\rho, v_1, v_2, v_3, H : \bar{\Omega} \to R_1$ such that they satisfy the following equations and conditions:

(1) $$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}(x_2 \rho v_i) = 0,$$

(2a) $$\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) v_1 = \frac{\partial H}{\partial x_1} - \frac{1}{2x_2^2}\frac{\partial (x_2 v_3)^2}{\partial x_1},$$

(2b) $$-\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) v_2 = \frac{\partial H}{\partial x_2} - \frac{1}{2x_2^2}\frac{\partial (x_2 v_3)^2}{\partial x_2},$$

(2c) $$0 = \sum_{i=1}^{2} v_i \frac{\partial (x_2 v_3)}{\partial x_i} \quad \text{in } \Omega;$$

(3) $\rho = \rho_0$ in Ω, if the fluid is incompressible,

(3*) $P(\rho) = H - \frac{1}{2}(v_1^2 + v_2^2 + v_3^2)$ in Ω for the compressible fluid

(4) $x_2 \rho (v_1, v_2) \cdot \vec{n} = \varphi$ on $\partial\Omega$.

Let $\Gamma_1 \subset C_0$ be an arc that determines the inlet, through which the fluid enters. Then

(5) $H|\Gamma_1 = h, \quad x_2 v_3|\Gamma_1 = w.$

If $k \geq 1$ (Ω is a multiply connected domain), then

(6) $v_1(x_i) = v_2(x_i) = 0, \quad x_i \in C_i, \quad i = 1, \dots, k.$

Notation: ρ - density, v_1, v_2, v_3 - velocity components in the cylindrical coordinates x_1, x_2, ε, H - total energy, $\vec{n}$ - unit outer normal to $\partial\Omega$. $\rho_0 > 0$ is a given constant, $P : (0, +\infty) \to R_1$ is a given increasing function, $\varphi \cdot \partial\Omega \to R_1$, $k, w : \Gamma_1 \to R_1$ are given functions, $x_i \in C_i$ are given points, the conditions (6) are the so called trailing conditions.

Additional assumptions: $\varphi|\Gamma_1 < 0$, $\int_{C_0} \varphi \, dS = 0$, and, if $\varkappa \geq 1$, $\varphi | C_1 \cup \dots \cup C_\varkappa = 0$.

2. Stream function

The solvability of the problem (P1) was studied with the use of the so called stream function ψ satisfying the relations

$$(7) \qquad \frac{\partial\psi}{\partial x_1} = -\rho x_2 v_2 , \qquad \frac{\partial\psi}{\partial x_2} = \rho x_2 v_1 .$$

For incompressible and subsonic compressible stream fields the problem (P1) was transformed into the following

Problem (P2). To find $\psi : \bar{\Omega} \to R_1$ and, if $\varkappa \geq 1$, $\psi = (\psi_1, \dots, \psi_\varkappa) \in R_\varkappa$ such that

$$(8) \quad \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(b(x, \psi, (\nabla\psi)^2) \frac{\partial\psi}{\partial x_i}\right) = f(x, \psi, (\nabla\psi)^2) \qquad \text{in } \Omega ,$$

$$(9) \quad \psi | C_0 = \psi_0$$

and, if $\varkappa \geq 1$,

$$(10) \quad \psi | C_i = \psi_i = \text{const} , \qquad i = 1, \dots, \varkappa ,$$

$$(11) \quad \frac{\partial\psi}{\partial n}(x_i) = 0 , \qquad i = 1, \dots, \varkappa .$$

The functions b, f are given by the relations (5) and either (3) or (3*). The function ψ_0 is obtained by integrating φ from the condition (4) along the curve C_0 ([1, 2]).

3. Solution of the problem

1) In the case $\varkappa = 0$ (Ω is simply connected) the problem (P2) was solved in the Sobolev space $W_2^1(\Omega)$ with the use of the monotone and pseudomonotone operators. After the application of known regularity results for elliptic problems we get the classical solvability of the problem (P2) and afterwards, the classical solvability of the general rotational, compressible flow problem (P1). The results are formulated in [1, 2] .

2) If $\varkappa \geq 1$ (Ω is multiply connected), the problem (P2) was studied theoretically and numerically in two cases:

a) Irrotational compressible flows. The equation (8) has the form

$$(8^*) \quad \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(b(x, (\nabla\psi)^2) \frac{\partial\psi}{\partial x_i}\right) = 0 ,$$

to which we add the conditions (9) - (11). This problem was transformed to a system of nonlinear algebraic equations and the solvability was proved on the basis of the strong maximum principle and appropriate apriori estimates (see [4]).

b) Rotational incompressible flows. The equation (8) has the form

$$(8^{**}) \qquad \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(b(x)\frac{\partial \psi}{\partial x_i}\right) = f(x, \psi).$$

The solvability of the problem (8^{**}), (9) - (11) was proved in [3] with the use of the strong maximum principle, apriori estimates and the Schauder fixed-point theorem.

4. Numerical solution

The problem (P2) was solved numerically by the finite-difference method. After the discretization of the differential equation (8^{*}) or (8^{**}), the boundary conditions (9), (10) and the conditions (11) we get the finite-difference, generally nonlinear system ([1]).

a) Incompressible flows. The finite-difference system has the form

$$(12) \qquad A\overline{\psi} = \Phi(\overline{\psi}).$$

$\overline{\psi} \in R_N$ is a vector whose components are the approximate values of the stream function at mesh points, A is an $N \times N$ irreducibly diagonally dominant matrix (IDDM), $(\Phi(\overline{\psi}))_i = \phi_i(\overline{\psi}_i)$, $\phi_i : R_1 \to R_1$, $i = 1, \dots, N$. The system (12) has at least one solution and was solved iteratively by the Newton-relaxation method: we write $A = A_L + D + A_U$; where the matrix A_L is strictly lower triangular, D - diagonal and A_U-strictly upper triangular, $\overline{\psi}^0 \in R_N$,

$$(13) \qquad A_L\overline{\psi}^{n+1} + (D - \Phi'(\overline{\psi}^n))\overline{\psi}^* + A_U\overline{\psi}^n = \Phi(\overline{\psi}^n) - \Phi'(\overline{\psi}^n)\overline{\psi}^n,$$

$$(14) \qquad \overline{\psi}^{n+1} = \overline{\psi}^n + \omega(\overline{\psi}^* - \overline{\psi}^n).$$

We use the relaxation parameter $\omega \in (0, 2)$. The method converges usually with $\omega = 1$.

b) Irrotational compressible flow. We get the finite-difference system

$$(15) \qquad A(\overline{\psi})\overline{\psi} = \Phi(\overline{\psi}).$$

$\Phi : R_N \to R_N$, $A(\overline{\psi})$ is IDDM for every $\overline{\psi} \in R_N$. (15) has a unique solution, which was found by the relaxation method, given by the formulae

$$(16) \qquad A_L(\overline{\psi}^n)\overline{\psi}^{n+1} + D(\overline{\psi}^n)\overline{\psi}^* + A_U(\overline{\psi}^n)\overline{\psi}^n = \Phi(\overline{\psi}^n)$$

and (14), where $\omega \in (0, 1\rangle$. A series of numerical experiments showed that the method converges safely with $\omega = 1$, if the sought stream field is subsonic.

References

[1] M. Feistauer: Mathematical solution of three-dimensional axially symmetric stream fields of an ideal fluid. Technical research report , ŠKODA Plzeň, 1978 (in Czech).

[2] M. Feistauer: Mathematical study of three-dimensional axially symmetric stream fields of an ideal fluid. Proceedings of the Conference "Methods and Techniques of Mathematical Physics", Oberwolfach 1980, P.D.Lang, Frankfurt am Main.

[3] M. Feistauer: Mathematical study of rotational incompressible non--viscous flows through multiply connected domains. To appear in Apl.mat., 26(1981).

[4] M. Feistauer: Subsonic irrotational flows in multiply connected domains. To appear in Math. Methods in the Applied Sciences, Vol. 4 (1982).