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TRANSFORMATIONS AND FACTORISATION OF ORDINARY NONLINEAR DIFFERENTIAL EQUATIONS

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In the constructive investigation of differential equations the factorisation and transformation methods play a great role. But their separate usage as a rule is not quite effective: there are no general rules for searching of substitutions and finding of the explicit form of factorisation faces principal difficulties.
In the report the advantages of using these methods together are shown. It gives an opportunity to expend them to the nonlinear case and to obtain concrete results concerned with integration and qualitative investigation of broad class of differential equations.

I.Factorisation and reducibility of linear equations. Consider a linear differential equation

$$Ly \equiv \sum_{k=0}^{n} a_k(x) y^{(k)} = 0, \; (') = d/dx, \; a_n = 1, \; a_k(x) \in C_I^k, \; I = (\alpha, \beta) \quad \text{(I)}.$$

According to the G.Mammana theorem there always exists factorisation /in general noncommutative/ in terms of the first order differential operators

$$Ly \equiv \prod_{k=n}^{1} (\mathcal{D} - \alpha_k(x)) y = 0, \quad \mathcal{D} = d/dx, \tag{2}$$

$\alpha_k(x)$ are generally complex-valued functions in x. There fore one fails to construct such factorisation in general case as it is necessary to solve the Lyapunov-Poincare $(n-I)$ order nonlinear differential equation

$$L \exp\left(\int \alpha_1 dx\right) \equiv \sum_{k=0}^{n} a_k(x) \left\{\exp\left(\int \alpha_1 dx\right)\right\}^{(k)} = c \tag{3}$$

/when n=2 (3) becomes the Riccati equation /.
Effective explicite factorisation can be realised if one employes to its structure additional requicements. There is remarcable explicit factorisation structure for commutative multipliers $(\mathcal{D} - \alpha_k(x)), \; k = \overline{1,n}$ when $\alpha_k = -\frac{1}{n} a_{n-1} + \lambda_k, \; \lambda_k = \text{const}$, and (2) is reducible by the transformation $y = \exp\left(-\frac{1}{n}\int a_{n-1} dx\right) z$ to the equation with constant coefficients.
More general is the class of differential equation being reducible by the transformation

$$y = v(x) z, \quad dt = u(x) dx; \quad u, v \in C_I^k, \quad uv \neq 0 \tag{4}$$

to the equation

$$Mz \equiv \sum_{k=0}^{n} b_k \overset{(k)}{z} = 0, \quad b_k = const, \quad (\dot{}) = d/dt. \qquad (5)$$

Theorem I. In order to reduce (I) to (5) by (4) it is necessary and sufficient that either the factorisation /noncommutative/

$$Ly \equiv \prod_{k=n}^{1} \left[D - \frac{v'}{v} - (k-1)\frac{u'}{u} - \tau_k u \right] y$$

or the commutative factorisation

$$\frac{1}{u^n} Ly \equiv \prod_{k=1}^{n} \left(\frac{1}{u} D - \frac{v'}{vu} - \tau_k \right) y$$

where τ_k are the roots of the characteristic equation $\sum_{k=0}^{n} b_k \tau^k = 0$ is hold.

In this case it is necessary / and for n=2 at the same time sufficient / that

$$v(x) = |u(x)|^{\frac{1-n}{2}} \exp\left(-\frac{1}{n} \int a_{n-1} dx + \frac{1}{n} b_{n-1} \int u dx \right),$$

$$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 + \frac{6B_{n-2}}{n(n^2-1)} u^2 = \frac{6A_{n-2}}{n(n^2-1)},$$

where

$$A_{n-2} = a_{n-2} - \frac{n-1}{2n} a_{n-1}^2 - \frac{n-1}{2} a'_{n-1},$$

$$B_{n-2} = b_{n-2} - \frac{n-1}{2n} b_{n-1}^2$$

are semiinvariants of (I) and (5) with respect to the transformation of the function only $y = v(x) z$.

There is the differential relation between v and u:

$$\sum_{k=0}^{n} a_k(x) v^{(k)} - b_0 u^n v = 0.$$

3. Factorisation of nonlinear reducible equations.

Theorem 2. In order to reduce the nonlinear nonautonomous n-order differential equation

$$f(x, y, y', \ldots, y^{(n)}) = 0 \qquad (6)$$

to (5) by the transformation

$$y = v(x, y) z, \quad dt = u(x, y) dx \qquad (7)$$

it is necessary and sufficient that

$$f \sim \prod_{k=n}^{1} \left[D - \frac{v_x + v_y y'}{v} - (k-1)\frac{u_x + u_y y'}{u} - \tau_k u \right] y \qquad (8)$$

$v_x = \partial v/\partial x$ are the roots of $\sum_{k=0}^{n} b_k \tau^k = 0$.

The important special cases of (7) are the transformation (4) carrying out autonomisation /elimination of independent variable from the equation / and transformation

$$y = v(y) z \quad , \quad dt = u(y)\, dx \tag{9}$$

carrying out exact linearisation of equations.

4. The autonomisation gives an opportunity to find the conditions for reduction 6 to autonomous form and to solve an inverse problem of finding the form of variable coefficients necessary for this, and also to obtain exact solutions.

Lemma I. In order to reduce (6) to the autonomous form

$$\varphi(z, \dot z, \dots, \overset{(n)}{z}) = 0 \tag{I0}$$

by (4) it is necessary and sufficient (6) admits the one-parameter Lie group with generator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

and has the following structure

$$X = \frac{1}{u}\frac{\partial}{\partial x} + \frac{v'}{uv} y \frac{\partial}{\partial y}. \tag{II}$$

In the theory of differential equations and numerous applications there are often the equations represented as a sum of linear and non-linear parts which are suitable to be written in form

$$y'' + a_0(x) y + f(x) y^{\nu} F(x, y, y') = 0. \tag{I2}$$

Theorem 1. For reduction of (I2) to the autonomous form

$$\ddot z \pm b_1 \dot z + b_0 z + a z^{\nu} \Phi(z, \dot z) = 0, \tag{I3}$$

it is necessary and sufficient that

$$f(x) = a v^{1-\nu} u^2, \quad F(x, y, y') = \Phi\left[\frac{y}{v}, \left(\frac{1}{u}D - \frac{v'}{vu}\right) y\right],$$

where $v(x) = |u|^{-1/2} \exp(\pm \frac{1}{2} b_1 \int u\, dx)$ and $u(x)$ satisfies the Kummer–Schwarz second order equation

$$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 + \left(b_0 - \frac{1}{4}b_1^2\right) u^2 = a_0(x),$$

in this case (I2) has the exact particular solution of the form $y = \rho v(x)$ where ρ is a root of $b_0 \rho + a \rho^{\nu} \Phi(\rho, 0) = 0$. For reduction (I2) to (I3) by (4) it is necessary and sufficient $f(x)$ satisfies

$$f'' - \frac{\nu+4}{\nu+3}\frac{f'^2}{f} - (\nu+3)\mathcal{I}_k(x) f + \frac{[\ldots(\nu+3) - 2b_1^2(1+\nu)/(3+\nu)]f^{\frac{\nu+7}{\nu+3}}}{k + \frac{b_1^2(1-\nu)^2}{(3+\nu)^2}\left(\int f^{\frac{2}{\nu+3}}dx\right)} = 0$$

/k=I, if b_I=0_; k=0, if b_I=0/.

5.Exact linearisation gives opportunity to find a general form of the nonlinear autonomous equation $F(y, y', \ldots, y^{(n)}) = 0$ admitting an exact reduction to (5) by the nonlinear transformation (9), to construct invariants /the first integrals/, to obtain the general solution of the nonlinear equation in a parametric form, to investigate the equations of forced nonlinear oscillations of the forme $F(y, y', \ldots, y^{(n)}) = \varphi(x)$. In particular the second order equation which can be linearised must have structure

$$y'' + f_2(y) y'^2 + b_1 f_1(y) y' + f_0(y) = 0, \qquad \text{(I4)}$$

where $f_1(y)$, $f_2(y)$ are orbitrary functions, $f_0 = f_1 \exp(-\int f_2 dy)$ or

$$f_2 = 2a/(ay+b) - (f_1)_y/f_1, \quad f_0 = f_1^2 (ay+b) b_0 y / b.$$

In these cases (I4) has the general solution respectively

$$y = \chi^{-1}(Z(t)), \quad \chi(y) = \beta \int f_1 \exp\left(\int f_2 dy\right) dy, \quad x = \int \frac{dt}{f_1(y(t))};$$

$$y = \frac{bZ(t)}{1 - aZ(t)}, \quad x = \int \frac{dt}{f_1(y(t))}$$

and admits the first integrals /invariants/ I_k:

$$I_{1k}(x) = \tau_k x - \beta \int \frac{\exp\left(\int f_2 dy\right)}{\chi(y)} dy, \quad I_{2k}(x) = \tau_k x - b \int \frac{dy}{y(ay+b) f_1(y)}, \quad k=1,2.$$

The analogous results have been obtained for the equations of the third and higher orders but because of its complication we shall not take them here.

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