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SOLVABILITY PROPERTIES OF SEMILINEAR OPERATOR EQUATIONS

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1. Introduction

We report on some recent results on the solvability of nonlinear operator equations of the form

$$(1) \qquad Au = F(u),$$

where $A : \mathrm{dom}(A) \subset H \to H$ is a self-adjoint linear operator in the real Hilbert space $H = (H,\langle\cdot,\cdot\rangle)$, and F is an appropriately defined nonlinear potential operator. Equations of this type can be considered as abstract formulations for a variety of problems in differential equations. For simplicity we restrict ourselves to the following three characteristic examples (E), (W), and (H):

Find solutions u of the elliptic boundary value problem

$$\text{(E)} \qquad -\Delta u = f(x,u) \quad \text{in } \Omega, \quad u = o \quad \text{on } \partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$.

The second example concernes the existence of 2π-*periodic* solutions of the nonlinear wave equation

$$\text{(W)} \qquad \begin{aligned} &u_{tt} - u_{xx} = f(x,t,u) \quad \text{in} \quad (o,\pi) \times \mathbb{R}, \\ &u(o,t) = u(\pi,t) = o \qquad \forall\, t \in \mathbb{R}. \end{aligned}$$

Finally, in the last example we are interested in the existence of 2π-*periodic* solutions of the Hamiltonian system

$$\text{(H)} \qquad \dot{p} = -H_q(p,q,t), \ \dot{q} = H_p(p,q,t), \quad p,q \in \mathbb{R}^n.$$

In each case we assume that the nonlinearities are given smooth functions, and we refer to the original research papers for more general cases.

2. Uniqueness Results

Experience tells that one should expect unique solvability of equation (1) if the nonlinearity does not "interact" with the spectrum $\sigma(A)$ of the linear operator. The following general existence and uniqueness theorem confirms this fact.

THEOREM 1: *Suppose that* $F : H \to H$ *is continuous and satisfies*

$$(2) \qquad \nu \leq \frac{<F(u)-F(v),u-v>}{\|u-v\|^2} \leq \mu \qquad \forall\ u,v \in H,\ u \neq v,$$

where $\nu \leq \mu$ *are real numbers such that* $[\nu,\mu] \subset \rho(A) := \mathbb{R}\backslash\sigma(A)$. *Then the equation* $Au = F(u)$ *has exacty one solution.*

It should be observed that there is no condition whatsoever concerning the nature of the spectrum of A outside the interval $[\nu,\mu]$. Theorem 1 requires only the existence of a gap in the spectrum such that the nonlinearity "varies only in this gap".

Theorem 1 has been obtained in [1] (cf. [1, Theorem 3.4]) as a simple by-product of general considerations about saddle points of nonlinear functions on Hilbert spaces. Recently J. Mawhin [17] has given a direct proof of this simple result, which generalizes and unifies numerous earlier existence and uniqueness theorems.

The above theorem has obvious and easy applications to our model problems (E) and (W). However, it is too restrictive for applications to problem (H) or to *systems* of elliptic boundary value problems or nonlinear wave equations. In this case the linearization $F'(u)$ of F is represented by a matrix. If the eigenvalues of this matrix lie in distinct gaps of $\sigma(A)$, then the nonlinearity does also not interact with the spectrum of A, but condition (2) is not satisfied. The following theorem takes care of this case.

<u>THEOREM 2</u>: *Suppose that* $H = L^2(\Omega,\mathbb{R}^M)$ *for some* σ*-finite measure space* Ω *and* $M \geq 1$. *Let* $f : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ *be a Carathéodory function such that* $f(\omega,.)$ *is continuously differentiable with a symmetric derivative* $D_2f(\omega,\xi) \in L(\mathbb{R}^M)$. *Suppose that there exist symmetric matrices* $b^{\pm} \in L(\mathbb{R}^M)$ *such that*

$$b^- \leq D_2f(\omega,\xi) \leq b^+ \qquad \forall\ (\omega,\xi) \in \Omega \times \mathbb{R}^M,$$

and denote by $\lambda_1^{\pm} \leq \ldots \leq \lambda_M^{\pm}$ *the eigenvalues of* $b^{\pm}$. *Finally suppose that* A *commutes with every multiplication operator induced by a constant symmetric matrix, and that* A *has a pure point spectrum in* $[\lambda_1^-,\lambda_M^+]$. *Then the equation*

$$Au = f(\omega,u) \quad \text{in } \Omega$$

has exactly one solution provided

$$\bigcup_{j=1}^{M} [\lambda_j^-,\lambda_j^+] \subset \rho(A).$$

The assumption that A has a *pure* point spectrum in $[\lambda_1^-,\lambda_M^+]$

means that the closure of the subspace $(E_{\lambda_M^+} - E_{\lambda_1^-})(H)$ has an orthonormal basis of eigenvectors of A, where $\{E_\lambda \mid \lambda \in \mathbb{R}\}$ is the spectral resolution of A. In particular, this is the case if $\sigma(A) \cap [\lambda_1^-, \lambda_M^+]$ consists of finitely many eigenvalues of *arbitrary* multiplicities. For this reason Theorem 2 is applicable to *systems of nonlinear wave equations in* n *space dimensions*. In this case $\sigma(A) = \sigma_p(A)$, but in general, there are infinitely many eigenvalues of infinite multiplicities (for example, if Ω is the n-dimensional cube $(o,\pi)^n$).

Theorem 2 is a special case of a more general abstract theorem for equations of the form (1) in arbitrary Hilbert spaces (cf. [2, Theorem 2.1]). Its proof is based on the theory of monotone operators in conjunction with a Galerkin type argument. The results in [2] generalize considerably earlier results of J. Mawhin [18], Brown and Lin [10], Bates [8], Bates and Castro [9], and others.

3. Asymptotically Linear Problems

We turn now to the question of the existence of multiple solutions, where we consider first the case that $F \in C(H,H)$. If f is the Nemytskii operator of a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ on $H = L^2(\Omega)$, then it is well known that this is the case iff f is linearly bounded.

For simplicity of presentation we consider only problems (E) and (W). Moreover, we suppose that f is independent of x, or (x,t), respectively, and that f is asymptotically linear, that is,

$$f'(\infty) = \lim_{|\xi| \to \infty} f'(\xi)$$

exists in $\mathbb{R}$. Finally in problem (W) we assume that $f'(\xi) \neq o$ for all $\xi \in \mathbb{R}$. Then the following theorem guarantees the existence of at least one nontrivial solution.

THEOREM 3: *Suppose that* $f(o) = o$ *and* $f'(\infty) \notin \sigma(A)$. *Then problem* (E) *or problem* (W), *resp., has at least one nontrivial solution provided there exists an eigenvalue* λ *of* A *such that*

$$\min\{f'(o), f'(\infty)\} < \lambda < \max\{f'(o), f'(\infty)\}.$$

This theorem is a special case of an existence theorem for the abstract equation (1) proved in [3, Theorem 9.4]. Its proof is based on a saddle point reduction introduced in [1] and on a subsequent application of the generalized Morse theory due to C.C. Conley [13]. Recently it has been shown by K.-C. Chang [12], that the generalized

Morse theory can be replaced by classical Morse theory on manifolds with boundary as established in [19].

Theorem 3 guarantees the existence of at least one nontrivial solution if $f'(\xi)$ crosses at least one eigenvalue when $|\xi|$ varies from o to ∞. We emphazise the fact that no further condition for the nonlinearity is necessary. Thus this theorem seems to be the most general and natural result in this direction, available so far.

Of course, Theorem 3 suggests the question: are there k nontrivial solutions if $f'(\xi)$ crosses k distinct eigenvalues of A? In the case of problem (E) this is not known. For problem (W) we have the following

THEOREM 4: *Suppose that* $f(o) = o$, $f'(\infty) \notin \sigma(A)$, *and* $f' \leq \beta < o$. *Moreover, suppose that, for some* $k \in \mathbb{Z}$ *and* $\ell \in \mathbb{N}$,

$$\min\{f'(o),f'(\infty)\} < \lambda_{k+1} < \dots < \lambda_{k+\ell} < \max\{f'(o),f'(\infty)\},$$

where $\dots \lambda_{-2} < \lambda_{-1} < \lambda_o = o < \lambda_1 < \lambda_2 < \dots$ *are the eigenvalues of the wave operator under the above periodicity conditions. Then problem* (W) *has at least* ℓ *nontrivial solutions.*

Whereas Theorem 3 holds also if f depends on (x,t), it is essential in Theorem 4 that f is independent of t, that is, that the wave equation is autonomous (in time). The proof of Theorem 4, which is given in [5], is based on the fact that the corresponding equation (1) is equivariant with respect to an appropriate action of the circle group S^1. Then the multiplicity result is essentially derived by Lusternik-Schnirelmann arguments on the orbit space $(E-\{o\})/S^1$, where E is an appropriate finite dimensional subspace of H, obtained by the saddle point reduction of [1].

Similar techniques apply to problem (H). Since the results corresponding to Theorems 3 and 4 are somewhat more complicated to state, we refer to [4].

In the above considerations we have assumed that $f'(\infty) \notin \sigma(A)$, that is, that there is no resonance at infinity. This condition guarantees that appropriate functionals satisfy the Palais-Smale condition. There are also multiplicity results in the case of "strong resonance", for example, if $f'(\xi) = \lambda \in \sigma(A)$ for $|\xi| \geq \xi_o > o$, e.g. [22,23]. Recently G. Cerami [11] has given an extension of the Palais-Smale condition, which can be used to handle some problems of strong resonance (cf. [7]).

The above theorems require the condition $f(o) = o$. Thus, in particular, they don't apply to equations of the form

$$u_{tt} - u_{xx} = f(u) + h$$

with $h \in L^2((o,\pi) \times (o,2\pi))$ and $h \neq o$. For some recent results in this direction we refer to [15]. However, it doesn't seem that there are definite results for these problems so far.

4. Superlinear Problems

In this final section we add a few remarks on problem (E) in the superlinear case, that is, if

$$\lim_{|\xi|\to\infty} \frac{f(x,\xi)}{\xi} = \infty.$$

Due to an example of Pohozaev [20], it is known that one cannot expect a solution to (E) in general if f grows too fast. In fact, in order to associate a well defined functional on $H_o^1(\Omega)$ with problem (E), which satisfies the Palais-Smale condition, it is known that one has to impose the growth restriction

$$|f(x,\xi)| \leq \alpha\, |\xi|^p + b(x) \qquad \forall\ (x,\xi) \in \Omega \times \mathbb{R}^n,$$

where

$$p < p^* := \frac{n+2}{n-2} \qquad \text{if} \quad n \geq 3.$$

If $n = 2$, then one can admit a growth condition of the form

$$|f(x,\xi)| \leq \alpha\, e^{\beta(\xi)} + b(x),$$

where $\beta(\xi) = o(\xi^2)$ as $|\xi| \to \infty$.

Thus, a typical superlinear boundary value problem would be:

$$(3) \qquad \begin{aligned} -\Delta u &= |u|^{p-1}u + h(x) \quad && \text{in} \quad \Omega, \\ u &= o && \text{on} \quad \Omega. \end{aligned}$$

In this case the nonlinearity interacts with infinitely many eigenvalues of the Laplacian and one can expect that problem (3) has infinitely many solutions (in particular since very general results of this type are true if $n = 1$; e.g. [14]).

Recently it has been shown by M. Struwe [21] and, independently, by A. Bahri and H. Berestyckii [6] that (3) has in fact infinitely many solutions for every $h \in L^2(\Omega)$, provided $1 < p < p_n$, where p_n is the largest root of $(2n-2)p^2 - (n+2)p - n = o$. Thus, $p_2 = 1 + \sqrt{2} \sim 2.4$ and $p_3 \sim 1.6$. The proof of this result is based on perturbation arguments for even functionals which are inspired by an early result of M.A. Krasnosel'skii [16]. If $p_n \leq p < p^*$, then it is not even known that (3) has at least one solution for every $h \in L^2(\Omega)$. (In [6] it is mentioned that Bahri has shown that (3) is solvable for every h in a dense subset of $L^2(\Omega)$.)

The methods of [6] and [21] use heavily the fact that the nonlinearity has near $+\infty$ the same asymptotic behavior as near $-\infty$. Thus, for example, nothing is known about the solvability of (E) if f is of the form

$$f(x,\xi) = \begin{cases} \xi^4 + h(x) & \text{for } \xi \geq o \\ -\xi^2 + h(x) & \text{for } \xi \leq o \end{cases}$$

for $n \leq 3$.

In summary, there has much work to be done in order to understand the qualitative behavior of problem (1).

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