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Mapping properties of regular and strongly degenerate elliptic differential operators in the Besov spaces $B^s_{p,p}(\Omega)$. The case $0 < p < \infty$

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MAPPING PROPERTIES OF REGULAR AND STRONGLY DEGENERATE ELLIPTIC DIFFERENTIAL OPERATORS IN THE BESOV SPACES $B^s_{p,p}(\Omega)$. THE CASE $0<p<\infty$

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1. Main Results

Let Ω be a bounded C^∞-domain in the Euclidean n-space R_n. Let A,

$$(Af)(x) = \sum_{|\alpha|\leq 2m} a_\alpha(x)\, D^\alpha f(x), \qquad a_\alpha(x) \in C^\infty(\bar{\Omega}),$$

be a properly elliptic differential operator of order 2m. Here m = 1,2,3,... Let B_j,

$$(B_j f)(x) = \sum_{|\alpha|\leq m_j} b_{j,\alpha}(x)\, D^\alpha f(x), \qquad b_{j,\alpha}(x) \in C^\infty(\partial\Omega),$$

j = 1,..., m, be m differential operators defined on the boundary $\partial\Omega$ of Ω. All functions in this paper, in particular the coefficients of the above differential operators, are complex-valued. As usual, $\{A, B_1, \dots, B_m\}$ is said to be a regular elliptic problem if $0 \leq m_1 < m_2 < \dots < m_m \leq 2m-1$ and if $\{B_j\}_{j=1}^m$ is a normal system satisfying the complementing condition with respect to A. For details concerning these well-known definitions we refer to [1] (cf. also [4], pp. 361 - 363). It is convenient for our purpose to assume that the following additional assumption is satisfied.

Hypothesis. If $f(x) \in C^\infty(\bar{\Omega})$ such that $(Af)(x) = 0$ for $x \in \bar{\Omega}$ and $(B_j f)(x) = 0$ for $x \in \partial\Omega$ and $j = 1,\dots,m$, then $f(x) \equiv 0$ in $\bar{\Omega}$.

Remark 1. In other words, it is assumed that the origin belongs to the resolvent set if $\{A, B_1,\dots, B_m\}$ is considered as a mapping between appropriate function spaces.

Definition 1. (i) If

$$(1) \quad \begin{cases} \text{either } 1<p<\infty & \text{and } 0<s<\frac{1}{p} \\ \text{or } \quad 0<p\leq 1 & \text{and } n(\frac{1}{p}-1)<s<1 \end{cases}$$

then $B^s_{p,p}(\Omega)$ is the completion of $C^\infty(\bar{\Omega})$ in the quasi-norm (norm if $p \geq 1$)

$$(2) \quad \|f\|_{B^s_{p,p}(\Omega)} = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} dx\, dy\right)^{\frac{1}{p}}$$

(ii) If p and s satisfy (1) and if $m = 1,2,3,\dots$, then

(3) $B^{s+2m}_{p,p}(\Omega) = \{f \mid D^\alpha f \in B^s_{p,p}(\Omega)$ for all α with $|\alpha| \leq 2m\}$.

Remark 2. $B^{s+2m}_{p,p}(\Omega)$ is equipped in the usual way with a quasi-norm.

Remark 3. These are the underlying Besov spaces. The theory described below can be extended to an essentially larger class of Besov spaces $B^s_{p,q}(\Omega)$ and probably also to spaces of Hardy - Sobolev type defined on domains, cf. [8] . However the definitions are more complicated, cf. also Section 2. Furthermore, one can also include the case $p = \infty$, which yields as a special case the famous Agmon-Douglis-Nirenberg theory in the Hölder - Zygmund spaces $\mathcal{C}^s(\Omega) = B^s_{\infty,\infty}(\Omega)$, where $s > 0$, cf. [8] .

Definition 2. Let ν be the (outer) normal with respect to $\partial\Omega$. If p, s, and m have the meaning of Definition 1(ii) and if $k = 0,\dots, 2m-1$, then $B^{s+2m-k-\frac{1}{p}}_{p,p}(\partial\Omega)$ is the set of all distributions f on the compact C^∞ - manifold $\partial\Omega$ for which there exists a function $g \in B^{s+2m}_{p,p}(\Omega)$ with $\frac{\partial^k g}{\partial \nu^k}\Big|_{\partial\Omega} = f$.

Remark 4. The spaces $B^s_{p,p}(R_n)$ (and more general $B^s_{p,q}(R_n)$) can be defined for all values of s, p (and q) with $-\infty < s < \infty$, $0 < p \leq \infty$ (and $0 < q \leq \infty$). Using the standard method of local coordinates one can give a direct definition of the corresponding spaces on $\partial\Omega$, cf. [8] . In particular, the spaces in Definition 2 depend only on the difference 2m-k and not on the special choice of m and k. If $1 < p < \infty$, then one has a well-known assertion, cf. e. g. [4] , p. 330, (cf. also Step 4 in Section 2, where further comments, also concerning the correctness of Definition 2, are given).

Theorem 1. Let $\{A, B_1,\dots, B_m\}$ be regular elliptic and let the Hypothesis be satisfied. If p and s are given by (1) then $\{A, B_1, \dots, B_m\}$ yields an isomorphic mapping from

(4) $B^{s+2m}_{p,p}(\Omega)$ onto $B^s_{p,p}(\Omega) \times \prod_{j=1}^{m} B^{s+2m-m_j-\frac{1}{p}}_{p,p}(\partial\Omega)$.

Remark 5. The proof of this theorem is long and complicated. However in Section 2 we shall try to describe some of the main ideas and key-assertions of the proof. A more detailed version, including also more general spaces, will be published elsewhere, cf. [8].

In [4] , Chapter 6, we considered a rather general class of strongly degenerate elliptic differential operators in the framework of an L_p-theory, where $1 < p < \infty$. On the one hand, we want to extend this theory to the spaces $B^{s}_{p,p}$ in the sense of Definition 1(i), on the other hand, in order to avoid technical difficulties, we restrict ourselves to a model case. Again, Ω is a bounded C^{∞}-domain in R_n. The distance of $x \in \Omega$ from $\partial\Omega$ is denoted by $d(x)$.

Definition 3. If (1) is satisfied, $m = 1,2,3,\ldots$ and $\nu > 2m$, then $B^{s+2m}_{p,p}(\Omega, d^{-p\nu}(x))$ is the completion of $C^{\infty}_{0}(\Omega)$ in the quasi-norm (norm if $p \geq 1$)

$$(5) \qquad \|f\|_{B^{s+2m}_{p,p}(\Omega)} + \|d^{-\nu} f\|_{B^{s}_{p,p}(\Omega)} .$$

Theorem 2. If all the parameters have the same meaning as in Definition 3 and if λ is a complex number with sufficiently large real part, then the operator $A + \lambda E$,

$$(6) \qquad (Af)(x) = (-\Delta)^{m} f + d^{-\nu}(x)\, f(x), \qquad E \text{ identity,}$$

yields an isomorphic mapping from $B^{s+2m}_{p,p}(\Omega, d^{-p\nu}(x))$ onto $B^{s}_{p,p}(\Omega)$.

Remark 6. In Section 3 we sketch some main ideas of the proof. Theorem 2 can be extended essentially to more general operators and also to a wider class of underlying spaces. Detailed proofs and a precise description of the mentioned extensions will be published elsewhere, cf. [7] .

2. Outline of the Proof of Theorem 1

Step 1. (Extension). If p and s satisfy (1) and if Ω in (2) and (3) is replaced by R_n, then one obtains corresponding spaces $B^{s+2m}_{p,p}(R_n)$. First of all we need properties of the spaces $B^{s+2m}_{p,p}(\Omega)$ and $B^{s+2m}_{p,p}(R_n)$. It can be shown that $B^{s+2m}_{p,p}(\Omega)$ is the restriction of $B^{s+2m}_{p,p}(R_n)$ to Ω (factor space) and that there exists a linear and bounded extension operator from $B^{s+2m}_{p,p}(\Omega)$ into $B^{s+2m}_{p,p}(R_n)$.

Step 2. (Fourier decomposition and Fourier multiplier). By Step 1 it is clear that properties of the spaces $B^{\sigma}_{p,q}(R_n)$ (in our case $\sigma = s + 2m$ and $p = q$ with the above restrictions) are of interest. Peetre's definition of the Besov spaces $B^{\sigma}_{p,q}(R_n)$ with $-\infty < \sigma < \infty$, $0 < p \leq \infty$, and $0 < q \leq \infty$ is the following. Let $S(R_n)$ be the Schwartz space and let $S'(R_n)$ be the space of tempered distributions. Let $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty} \subset S(R_n)$ be a smooth dyadic resolution of unity in R_n, i. e. $0 \leq \varphi_j(x) \leq 1$, $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ for $x \in R_n$,

$$\operatorname{supp} \varphi_0 \subset \{y \mid |y| \leq 2\}, \quad \operatorname{supp} \varphi_j \subset \{y \mid 2^{j-1} \leq |y| \leq 2^{j+1}\}$$

if $j = 1,2,\ldots$; for any multi-index γ there exists a constant c_γ such that

$$|D^{\gamma}\varphi_j(x)| \leqq c_\gamma\, 2^{-j|\gamma|}, \quad x \in R_n, \quad j = 0,1,2,\ldots$$

If $-\infty < \sigma < \infty$, $0 < p \leqq \infty$, and $0 < q \leqq \infty$, then

$$B^{\sigma}_{p,q}(R_n) = \{f \mid f \in S'(R_n),\ \|f\|^{\varphi}_{B^{\sigma}_{p,q}(R_n)} = \Big[\sum_{j=0}^{\infty} 2^{j\sigma q}\Big(\int_{R_n} |F^{-1}[\varphi_j Ff](x)|^p dx\Big)^{\frac{q}{p}}\Big]^{\frac{1}{q}} < \infty$$

for all systems $\varphi\}$,

(usual modification if p or q equals ∞). Here F and F^{-1} are the Fourier transform and its inverse on R_n, respectively. It can be shown that $B^{\sigma}_{p,q}(R_n)$ is a quasi-Banach space, where all the quasi-norms $\|f\|^{\varphi}_{B^{\sigma}_{p,q}(R_n)}$ for different choices of φ are mutually equivalent. Furthermore, $B^{\sigma}_{p,q}(R_n)$ coincides with the above spaces $B^{s+2m}_{p,p}(R_n)$ if $\sigma = s+2m$ and $p = q$ (under the above restrictions of the parameters s, p and m). All the spaces $B^{\sigma}_{p,q}(R_n)$ satisfy the following weak Michlin - Hörmander Fourier multiplier property. There exists a natural number M and a positive number c (depending on σ, p and q) such that for all infinitely differentiable functions m(x) on R_n and all $f \in B^{\sigma}_{p,q}(R_n)$

$$\|F^{-1}[m(\cdot)Ff]\|_{B^{\sigma}_{p,q}(R_n)} \leqq c\Big(\sup_{\substack{|\alpha| \leqq M \\ x \in R_n}} (1+|x|^2)^{\frac{|\alpha|}{2}} |D^{\alpha}m(x)|\Big) \|f\|_{B^{\sigma}_{p,q}(R_n)}.$$

(We omit the index φ in $\|\cdot\|^{\varphi}_{B^{\sigma}_{p,q}(R_n)}$ because all these quasi-norms are mutually equivalent). Proofs of the assertions in this step may be found in [3] and [5].

Step 3. (Properties of the spaces $B^{\sigma}_{p,q}(R_n)$). The goal is to extend Arkeryd's proof (cf. [2] or [4], Chapter 5) for boundary value problems of $\{A, B_1, \ldots, B_m\}$ in the framework of an L_p-theory with $1 < p < \infty$ to the spaces $B^{s}_{p,p}$ in the sense of Definition 1(i). For this purpose, beside the extension property and the Fourier multiplier property described in the preceding steps, some other properties of the corresponding spaces on R_n are indispensable. (i) (Diffeomorphic mappings, cf. [8]). If $y = \psi(x)$ is an infinitely differentiable one-to-one mapping from R_n onto itself such that $\psi(x) = x$ for large values of $|x|$, then $f(x) \longrightarrow f(\psi(x))$ yields an isomorphic mapping from $B^{\sigma}_{p,q}(R_n)$ onto itself. Here $-\infty < \sigma < \infty$, $0 < p \leqq \infty$, and $0 < q \leqq \infty$. (ii) (Multiplication property, cf. [5]). If $g(x) \in C^{\infty}_{0}(R_n)$ then $f(x) \longrightarrow g(x)\, f(x)$ yields a linear and bounded mapping from $B^{\sigma}_{p,q}(R_n)$ into itself. Again $-\infty < \sigma < \infty$, $0 < p \leqq \infty$ and $0 < q \leqq \infty$.

Step 4. (Spaces on domains and manifolds). The two properties described in Step 3 (diffeomorphic mappings and multiplication property) are the basis for the well-known method of local coordinates. This gives the possibility to define the spaces $B^{\sigma}_{p,q}(\partial\Omega)$, where $-\infty < \sigma < \infty$, $0 < p \leqq \infty$ and $0 < q \leqq \infty$ by standard arguments, cf. [4], pp. 280/81 for the usual Besov spaces. The next step shows that these spaces coincide with the corresponding spaces in Definition 2 (under the restrictions of the parameters in the sense of Definition 2). Finally, by restriction of $B^{\sigma}_{p,q}(R_n)$ to Ω one can define spaces $B^{\sigma}_{p,q}(\Omega)$ for all values $-\infty < \sigma < \infty$, $0 < p \leqq \infty$ and $0 < q \leqq \infty$. All these spaces have the extension property described in Step 1, cf. [8].

Step 5. (Traces). Let ν be the (outer) normal on $\partial\Omega$ and let $r = 0,1,2,\ldots$ By the above properties and the assertion in [5], 2.4.2, it follows that R,

$$Rf = \left\{ f\big|_{\partial\Omega}, \frac{\partial f}{\partial\nu}\Big|_{\partial\Omega}, \ldots, \frac{\partial^r f}{\partial\nu^r}\Big|_{\partial\Omega} \right\},$$

is a linear and bounded mapping from $B^{\sigma}_{p,q}(\Omega)$ onto

$$\prod_{j=0}^{r} B^{\sigma-\frac{1}{p}-j}_{p,q}(\partial\Omega),$$

if $0 < p \leqq \infty$, $0 < q \leqq \infty$, and $s > r + \frac{1}{p} + \max(0, (n-1)(\frac{1}{p}-1))$. Now it follows that Definition 2 is meaningful and that the spaces defined there coincide with the corresponding spaces in the sense of the preceding step.

Step 6. (A-priori estimate). If p and s satisfy (1) then there exist two positive constants c_1 and c_2 such that for all $f \in C^{\infty}(\overline{\Omega})$

$$\begin{aligned} c_1 \|f\|_{B^{s+2m}_{p,p}(\Omega)} &\leqq \|Af\|_{B^{s}_{p,p}(\Omega)} + \|f\|_{B^{s}_{p,p}(\Omega)} + \\ &+ \sum_{j=1}^{m} \|B_j f\|_{B^{s+2m-m_j-\frac{1}{p}}_{p,p}(\partial\Omega)} \leqq c_2 \|f\|_{B^{s+2m}_{p,p}(\Omega)}. \end{aligned} \tag{7}$$

Here $\{A, B_1, \ldots, B_m\}$ is regular elliptic. For the proof of (7) it is not necessary that the above Hypothesis is true. The idea is to carry over Arkeryd's proof, cf. [2], of a corresponding a-priori estimate in the framework of an L_p-theory with $1 < p < \infty$, to the above basic spaces $B^{s}_{p,p}(\Omega)$ instead of $L_p(\Omega)$. We use the version of Arkeryd's proof given in [4], pp. 364 - 378. An examination of that proof shows that many arguments can be carried over from L_p to $B^{s}_{p,p}$ if one uses the 5 main assertions for general Besov spaces mentioned above: extension properties (Step 1 and Step 4), Fourier multiplier properties (Step 2), diffeomorphic mappings (Step 3), multiplication properties (Step 3), and traces (Step 5). However there remain essentially two points which are trivial for L_p-spaces but non-trivial for $B^s_{p,p}$-spaces. (i) If p and s satisfy (1) then S,

$$(Sf)(x) = \begin{cases} f(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in R_n - \Omega \end{cases}$$

is a linear and bounded operator from $B^s_{p,p}(\Omega)$ into $B^s_{p,p}(R_n)$. This assertion follows from the method of local coordinates and the considerations in [5] , 2.6.4. Cf. also [8] , Proposition 3.5. (ii) Let $a_\alpha(x)$ be the coefficients of A and let K be a ball in R_n with the centre x^0 and the radius τ , where we assume $0 < \tau < 1$. If p and s satisfy (1), then there exists a constant c, which is independent of x^0 and τ such that for all $\psi \in C_0^\infty(K)$ and all $f \in B^s_{p,p}(\Omega)$

$$\sum_{|\alpha|=2m} \|(a_\alpha(x) - a_\alpha(x^0))\, D^\alpha(\psi f)\|_{B^s_{p,p}(\Omega)} \leq c\tau\, \|\psi f\|_{B^{s+2m}_{p,p}(\Omega)} + c\, \|\psi f\|_{B^{s+2m-1}_{p,p}(\Omega)} .$$

Using the method of local coordinates then this estimate follows from

$$\|(a_\alpha(x) - a_\alpha(x^0))\, \psi f\|_{B^s_{p,p}(R_n)} \leq c\tau\, \|\psi f\|_{B^s_{p,p}(R_n)} ,$$

where again c is independent of τ . This inequality coincides essentially with formula (52) in [6] . If one uses the special properties (i) and (ii) and the above-mentioned general properties for the spaces $B^s_{p,p}$ then one obtains (7) in the same way as in [4] , pp. 364 - 378, where L_p is replaced by $B^s_{p,p}$.

<u>Step 7.</u> (Proof of Theorem 1). If the Hypothesis is satisfied then the term $\|f\|_{B^s_{p,p}(\Omega)}$ in (7) can be omitted (standard arguments). Now, Theorem 1 is a consequence of the classical theory for $\{A, B_1, \ldots, B_m\}$ and (7).

3. <u>Outline of the Proof of Theorem 2</u>

<u>Step 1.</u> (Mappings in the nuclear space $C_0^\infty(\bar{\Omega})$). If

$$C_0^\infty(\bar{\Omega}) = \{f \mid f \in C_0^\infty(R_n),\ \ \text{supp } f \subset \bar{\Omega}\},$$

then $A + \lambda E$ with a sufficiently large real part of λ yields an isomorphic mapping from $C_0^\infty(\bar{\Omega})$ onto itself. This is a special case of Theorem 1 in [4] , p. 420.

<u>Step 2.</u> (Decomposition). Next we need some properties of the spaces $B^s_{p,p}(\Omega)$, where s and p satisfy (1). There exists a constant c such that for all $f \in C_0^\infty(\Omega)$

(8) $$\int_{\Omega} d^{-sp}(x)|f(x)|^p\,dx \leq c\Big(\int_{\Omega}|f(x)|^p dx + \int_{\Omega\times\Omega}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\Big)^{\frac{1}{p}}.$$

This is a fractional Hardy inequality. A proof of (8) for $1<p<\infty$ may be found in [4] , p. 259. Using this result, one can extend (8) with $1<p<\infty$ to all couples (s,p) satisfying (1). In particular,

(9) $$\Big(\int_{\Omega} d^{-sp}(x)|f(x)|^p\,dx + \int_{\Omega\times\Omega}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\Big)^{\frac{1}{p}}$$

is an equivalent quasi-norm in $B^s_{p,p}(\Omega)$. Now we have a situation which is similar to that one in [4] , Subsection 3.2.3 and Subsection 6.3.1. The decomposition methods developed there can be applied (however some non-trivial additional considerations are necessary, for details we refer to [7]). Let $K_{j,l} = \{x \mid |x-x_{j,l}| < \tau\, 2^{-j}\}$ be balls such that $x_{j,l} \in \{y \mid y\in\Omega\ ,\ 2^{-j-1} \leq d(y) \leq 2^{-j}\}$ if j = 1,2, 3,..., with a sufficiently small τ . It is assumed that $\Omega = \bigcup_{j=0}^{\infty}\bigcup_{l=1}^{N_j} K_{j,l}$ (modification for j = 0 or for small values of j if necessary). Let $\varphi = \{\varphi_{j,l}\}_{\substack{j=0,1,2,\ldots\\ l=1,\ldots,N_j}}$ be a smooth resolution of unity with respect to the balls $K_{j,l}$, i. e. if $x\in\Omega$ then

$$0 \leq \varphi_{j,l}(x),\quad \sum_{j=0}^{\infty}\sum_{l=1}^{N_j}\varphi_{j,l}(x) = 1,\quad \operatorname{supp}\varphi_{j,l} \subset K_{j,l}\,.$$

Furthermore, for any multi-index γ there exists a number c_γ such that

$$|D^\gamma\varphi_{j,l}(x)| \leq c_\gamma\, 2^{j|\gamma|} \quad \text{if } j = 0,1,2,\ldots \text{ and } l = 1,\ldots,N_j.$$

Now it can be proved that for any system φ with the indicated properties

(10) $$\Big(\sum_{j=0}^{\infty}\sum_{l=1}^{N_j}\|\varphi_{j,l}f\|^p_{B^s_{p,p}(R_n)}\Big)^{\frac{1}{p}}$$

is an equivalent quasi-norm in $B^s_{p,p}(\Omega)$ (here s and p satisfy (1)). Similarly one obtains that for the spaces $B^{s+2m}_{p,p}(\Omega, d^{-\nu p}(x))$ described in Definition 3 and formula (5)

(11) $$\Big(\sum_{j=0}^{\infty}\sum_{l=1}^{N_j}\|\varphi_{j,l}f\|^p_{B^{s+2m}_{p,p}(R_n)} + 2^{j\nu p}\|\varphi_{j,l}f\|^p_{B^s_{p,p}(R_n)}\Big)^{\frac{1}{p}}$$

is an equivalent quasi-norm.

Step 3. (A-priori estimate). Now we have a situation which is similar as in [4] , Section 6.3. The proofs given there can be extended to the above case, where (10) and (11) play a decisive role. Again some non-trivial modifications are necessary, cf. [7] , where details are given. One obtains the following a-priori estimate. If p and s satisfy (1) then there exists a real number λ_0 and two positive numbers c_1 and c_2 such that for all complex numbers λ with Re $\lambda \geqq \lambda_0$ and for all $f \in C_0^\infty(\bar{\Omega})$

$$c_1 \quad \|(A+\lambda E)f\|_{B^s_{p,p}(\Omega)} \leqq \|f\|_{B^{s+2m}_{p,p}(\Omega, d^{-\nu p}(x))} + |\lambda| \; \|f\|_{B^s_{p,p}(\Omega)}$$

$$\leqq c_2 \; \|(A+\lambda E)f\|_{B^s_{p,p}(\Omega)} \; .$$

Step 4. (Proof of Theorem 2). Since $C_0^\infty(\bar{\Omega})$ is dense in $B^s_{p,p}(\Omega)$ and dense in $B^{s+2m}_{p,p}(\Omega, d^{-\nu p}(x))$, Theorem 2 is an easy consequence of Step 1 and the a-priori estimate of the preceding step.

References

[1] S. Agmon. On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. Comm. Pure Appl. Math. 15 (1962), 119 - 147.

[2] L. Arkeryd. On the L^p estimates for elliptic boundary problems. Math. Scand. 19 (1966), 59 - 76.

[3] J. Peetre. New Thoughts on Besov Spaces. Duke Univ. Math. Series. I. Duke Univ., Durham, 1976.

[4] H. Triebel. Interpolation Theory, Function Spaces, Differential Operators. North - Holland Publ. Comp., Amsterdam, New York, Oxford, 1978.

[5] H. Triebel. Spaces of Besov - Hardy - Sobolev Type. Teubner - Texte Math., Teubner, Leipzig, 1978.

[6] H. Triebel. On the Dirichlet problem for regular elliptic differential operators of second order in the Besov spaces $B^s_{p,q}$. The case $0 < p \leqq \infty$. Proc. Conf. "Elliptische Differentialgleichungen" held in Rostock, Sept. 1977 (to appear).

[7] H. Triebel. Strongly degenerate elliptic differential operators in Besov spaces. The case $0 < p \leqq \infty$. Beiträge Anal. 13 (1978).

[8] H. Triebel. On Besov - Hardy - Sobolev spaces in domains and regular elliptic boundary value problems. The case $0 < p \leqq \infty$. (to appear)

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