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SOME PROBLEMS CONCERNING THE FUNCTIONAL DIFFERENTIAL EQUATIONS

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We consider the equation

$$(1) \qquad \dot{x}(t) = f(t,x_t) ,$$

where $\dot{x}(t)$ means the derivative of the vector function $x(t)$ at the point t. If $x(t)$ is a function defined on the interval $[t_o-h,T)$, where $h>0$, t_o, T are real numbers, then $x_t = x(t+\theta)$, $\theta\in[-h,0]$ for $t\in[t_o,T)$ as usual. Let us explain the meaning of the notation used in this paper. Let n be a natural number, R^n the n-dimensional vector space of the points $X = (x_1,x_2,\dots,x_n)$ with a suitable norm $|\cdot|$, $C = C([-h,0],R^n)$ the Banach space of all continuous functions ψ with the norm $\|\psi\| = \max\{|\psi(t)|, t\in[-h,0]\}$, $C_o = \{\Phi\in C : \Phi(0) = 0\}$ a subspace of C. Furthermore, let $B = B([t_o,T),R^n)$ be the Banach space of all functions continuous and bounded on $[t_o,T)$ with the uniform norm $\|u\|_u = \sup\{|u(t)|, t\in[t_o,T)\}$ and $B_o = \{u(t)\in B : u(t_o) = 0\}$ a subspace of B. Let $\Phi\in C_o$ be fixed. Then $B_\Phi = \{z(t) : [t_o-h,T)\longrightarrow R^n, z(t) = \Phi(t-t_o)$ for $t\in[t_o-h,t_o]$, $z(t) = u(t)$ for $t\in[t_o,T)$; $u(t)\in B_o\}$ is a complete metric space with the metric $\varrho(z_1,z_2) = \|u_1-u_2\|_u$, where $z_1(t) = u_1(t)$, $z_2(t) = u_2(t)$ for $t\in[t_o,T)$, $u_1(t),u_2(t)\in B_o$.

As usual, the initial problem for (1) is formulated as follows: For given $t_o\in R$, $\psi\in C$ find a function $x\in C([t_o-h,A),R^n)$ such that $x(t) = \psi(t-t_o)$ for $t\in[t_o-h,t_o]$ and $\dot{x}(t) = f(t,x_t)$ for $t\in[t_o,A)$. We shall denote this solution by $x(t,t_o,\psi)$ and say that it is given by (t_o,ψ). Because every $\psi\in C$ can be written as $\psi(t) = X_o+\Phi(t)$, where $X_o = \psi(0)$, $\Phi\in C_o$, we shall write $x(t,t_o,X_o+\Phi)$ to express that the solution x passes through the point X_o at $t=t_o$.

Now, the main problem we will discuss is the following:

(P) Let be given $T\in R$, $t_o<T$, $X_o,X_1\in R^n$. Find $\Phi\in C_o$ such that the solution $x(t,t_o,X_o+\Phi)$ exists on $[t_o,T)$ and $\lim x(t,t_o,X_o+\Phi) = X_1$ as $t\to T-$.

The function f is subjected to the following hypotheses:

(H_1) $f(t,\psi)$ is continuous on $[t_o,T)\times C$ and $\int_{t_o}^{T} |f(t,0)|\,dt = K<\infty$.

(H_2) There is a function $\beta(t)$ continuous on $[t_o, T)$ such that $|f(t,\psi_1) - f(t,\psi_2)| \le \beta(t)\|\psi_1 - \psi_2\|$ for every

$$\psi_1, \psi_2 \in C \quad \text{and} \quad t \in [t_o, T) \quad \text{and} \quad \int_{t_o}^{T} \beta(t)dt = k < 1 .$$

The hypotheses H_1 and H_2 being satisfied, we can conclude

Theorem 1 [1] . Let H_1 and H_2 be satisfied. Then the solution $x(t,t_o,\psi)$, $\psi \in C$ exists on $[t_o, T)$, is unique and $\lim x(t,t_o,\psi) = X_1 \in R^n$ as $t \to T-$.

Theorem 2 [1] . Let H_1 and H_2 with $k < 1/2$ be satisfied. Let be given $X_1 \in R^n$, $\Phi \in C_o$. Then there exists a unique $X_o \in R^n$ such that $\lim x(t,t_o,X_o+\Phi) = X_1$ as $t \to T-$.

Now we define a map $F(X_o,\Phi) : R^n \times C_o \to R^n$ by the relation $F(X_o,\Phi) = \lim x(t,t_o,X_o+\Phi) = X_1$ as $t \to T-$. The following theorem mentions some properties of this map.

Theorem 3 [1] . Let H_1 and H_2 with $k < 1/2$ be valid. Then

a) the map $F(X_o,\Phi)$, by fixed Φ, is a one-to-one map of R^n onto R^n ;

b) $F(X_o,\Phi)$ fulfils the Lipschitz condition:
$$|F(X_{o1},\Phi_1) - F(X_{o2},\Phi_2)| \le e^k |X_{o1} - X_{o2}| + (e^k - 1)\|\Phi_1 - \Phi_2\| .$$

Our problem (P) was partially solved in the papers [1], [2], [3] . In [1] we obtained some results of negative character, e.g. if H_1 and H_2 are valid and if $X_o, X_2 \in R^n$, $|X_o| + K \ne 0$ and $|X_1| > [|X_o| + K] \frac{a}{1-k}$, $0 < a < 1$, then there is no solution of the problem (P) for $\Phi \in C_o$, $\|\Phi\| < K\frac{a}{1-k}$.

For further purposes we need an estimation of $|x(t,t_o,X_o+\Phi)|$ and $\|x_t(t_o,X_o+\Phi)\|$. Let us use the notation $x(t) = x(t,t_o,X_o+\Phi)$. Then, $x_t(\theta)$, $\theta \in [t-h,t]$ being continuous, there exists $v \in [t-h,t]$ such that $|x(v)| = \|x_t\|$. Suppose that $t \ge t_o$. Then either $v \ge t_o$ or $v \in [t_o-h,t_o]$.

Let $v \ge t_o$. Then we get

$$\|x_t\| = |x(v)| = \left|X_o + \int_{t_o}^{v} f(s,x_s)ds\right| \le |X_o| + \int_{t_o}^{v} |f(s,0)|\, ds +$$
$$+ \int_{t_o}^{v} |f(s,x_s) - f(s,0)|\, ds \le |X_o| + K + \|\Phi\| + \int_{t_o}^{t} \beta(s)\|x_s\| ds .$$

If $v \in [t_o-h,t_o]$, we have
$$\|x_t\| = |x(v)| = |X_o + \Phi(v-t_o)| \le |X_o| + \|\Phi\| + K + \int_{t_o}^{t} \beta(s)\|x_s\| ds .$$

Thus, for $t \in [t_o, T)$ we have

$$\|x_t\| \le |X_o| + K + \|\Phi\| + \int_{t_o}^{t} \beta(s)\|x_s\| ds \quad .$$

The application of Gronwall-Bellman lemma gives

$$(2) \qquad \|x_t\| \ [|(X_o| + K + \|\Phi\|] \ \exp\ (\int_{t_o}^{t} \beta(s)ds), \quad t \in [t_o, T) \ ,$$

which implies

$$(3) \qquad |x(t)| \le \|x_t\| \le [|X_o| + K + \|\Phi\|] \ \exp\ (\int_{t_o}^{t} \beta(s)ds, \ t \in [t_o, T).$$

Let us turn our attention to the dependence of $F(X_o, \Phi)$ on Φ by fixed X_o. It follows from Theorem 3 that, if $\|\Phi_1 - \Phi_2\| = 0$, we have $F(X_o, \Phi_1) = F(X_o, \Phi_2)$. If $\|\Phi_1 - \Phi_2\| \neq 0$, it may happen that $F(X_o, \Phi_1) \neq F(X_o, \Phi_2)$, but also $F(X_o, \Phi_1) = F(X_o, \Phi_2)$. If the former case occurs, it influences both solutions $x(t, t_o, X_o + \Phi_1)$ and $x(t, t_o, X_o + \Phi_2)$. The following theorem holds.

<u>Theorem 4.</u> Let $\|\Phi_1 - \Phi_2\| \neq 0$ and $F(X_o, \Phi_1) = F(X_o, \Phi_2)$. Then either

a) $0 < \|x(t, t_o, X_o + \Phi_1) - x(t, t_o, X_o + \Phi_2)\|_u < \|\Phi_1 - \Phi_2\|$

or

b) $\|x(t, t_o, X_o + \Phi_1) - x(t, t_o, X_o + \Phi_2)\|_u = 0$.

Proof. The function $|H(t)| = |x(t, t_o, X_o + \Phi_1) - x(t, t_o, X_o + \Phi_2)|; \ t \ge t_o$ is nonnegative and $|H(t_o)| = 0 = |H(T)|$. Thus there exists $t_1 \in [t_o, T)$ such that $H(t_1) = \max\{|H(t)|, \ t \in [t_o, T)\}$. If $t_1 = t_o$, the second case (b) occurs. If $t_1 \in [t_o + h, T)$ and $H(t_1) \neq 0$, the hypothesis H_2 yields

$$(4) \qquad |\dot{H}(t)| \le \beta(t)\|x_t(t_o, X_o + \Phi_1) - x_t(t_o, X_o + \Phi_2)\| \le \beta(t)|H(t_1)| \ , \quad t \in [t_o + h, T) \ .$$

Hence we get

$$(5) \qquad |H(t_1)| = |\int_{t_1}^{T} \dot{H}(t)dt| \le \int_{t_1}^{T} |\dot{H}(t)| dt \le \int_{t_1}^{T} \beta(t)dt |H(t_1)| \le \le k|H(t_1)|.$$

Therefore $t_1 \bar{\in} [t_o + h, T)$. Suppose that $t_1 \in (t_o, t_o + h)$ and that $|H(t_1)| \ge \|\Phi_1 - \Phi_2\|$. In this case the inequalities (4) and (5) hold as well and the same reasoning as above gives a contradiction which completes the proof.

In the following let X_o be fixed. We are going to examine the properties of the set of images of the set $G = \{\Phi \in C_o : \|\Phi\| \leq r\}$, $r > 0$, by the map F. We shall denote this set of images by $F(X_o, G)$.

Theorem 5. Let H_1, H_2 and H_3 be valid with

(H_3) There exists a constant d, $0 < d \leq 1$, such that for any $X_o \in R^n$ and any $y_i \in B_o$, $i=1,2$ and any $\Phi_i \in C_o$, $i=1,2$ the inequality

$$d\|\Phi_1 - \Phi_2\| \leq \left|\int_{t_o}^{t_o+h} [f(s, X_o + z_{1s}) - f(s, X_o + z_{2s})] ds\right|$$

holds where $z_i(t) = \Phi_i(t - t_o)$ for $t \in [t_o - h, t_o]$, $z_i(t) = y_i(t)$ for $t \in [t_o, t_o + h]$, $i=1,2$.

Then the set $F(X_o, G)$ is bounded, closed and connected.

Proof. The boundedness of $F(X_o, G)$ follows immediately from (3) or from Theorem 3, (b). Consider now the set of solutions $S = \{x(t, t_o, X_o + \Phi), \Phi \in G\}$ on $[t_o, T)$. From (3) we have that these solutions are uniformly bounded on $[t_o, T)$ by $[|X_o| + K + r]e^k$. The same holds also for the set $\{x_s(t_o, X_o + \Phi), s \in [t_o, T)\}$ as follows from (2). Further, for $t, t' \in [t_o, T)$, $t < t'$ we have

$$|x(t', t_o, X_o + \Phi) - x(t, t_o, X_o + \Phi)| \leq \int_t^{t'} |f(s,0)| ds + \int_t^{t'} \beta(s)\|x_s\| ds .$$

Now, from the existence of $\int_{t_o}^{T} |f(s,0)| ds$ and $\int_{t_o}^{T} \beta(s) ds$ and from the uniform boundedness of $\|x_s\|$ we get the equicontinuity of the elements of S on $[t_o, T)$. Thus we may apply on S the theorem of Ascoli-d'Arzelà on every compact set from $[t_o, T)$. Suppose that $X_i \in F(X_o, G)$, $i=1,2,\dots$ and that $\lim X_i = Y$ as $i \to \infty$. We are going to show that $Y \in F(X_o, G)$. Let $\{x(t, t_o, X_o + \Phi_i), \Phi_i \in G\}$ be the sequence of solutions of (1) such that $\lim x(t, t_o, X_o + \Phi_i) = X_i$ as $t \to T-$, $i=1,2,\dots$. Applying the Ascoli-d'Arzela theorem we get that we can choose a subsequence $\{x(t, t_o, X_o + \Phi_{i_k}), \Phi_{i_k} \in G\}$ from $\{x(t, t_o, X_o + \Phi_i)\}$ which converges to a continuous function $u(t)$ uniformly on every closed subinterval of $[t_o, T)$. Let $\lim x(t, t_o, X_o + \Phi_{i_k}) = X_{i_k}$ as $t \to T-$. Evidently $\lim X_{i_k} = Y$ as $k \to \infty$. The solutions $x(t, t_o, X_o + \Phi_{i_k})$ satisfy the equations

$$x(t,t_o,X_o+\Phi_{i_k}) = X_{i_k} - \int_t^T f(s,x_s(t_o,X_o+\Phi_{i_k}))ds \ , \ k=1,2,\dots \ .$$

The application of Lebesgue's dominated convergence theorem gives

$$u(t) = Y - \int_t^T f(s,u_s)ds \qquad \text{for} \quad t\in[t_o+h,T) \ .$$

Thus, we have got that $u(t)$ satisfies (1) on $[t_o+h,T)$ and $\lim u(t) = Y$ as $t\to T-$. The problem which appears here is: How to ensure that $u(t)$ satisfies (1) on $[t_o ,T)$; if this is possible, to which function $\Phi\in C_o$ this solution will correspond ? The validity of H_3 represents one of the possibilities. In fact, we know that the sequence $\{x(t_o+h,t_o,X_o+\Phi_{i_k})\}$ converges to $u(t_o+h)$. Therefore it is a Cauchy sequence. Using the hypothesis H_3, we get

$$|x(t_o+h,t_o,X_o+\Phi_{i_m}) - x(t_o+h,t_o,X_o+\Phi_{i_n})| =$$
$$= \left|\int_{t_o}^{t_o+h}[f(s,x_s(t_o,X_o+\Phi_{i_m}))-f(s,x_s(t_o,X_o+\Phi_{i_n}))]ds\right| \geq d\|\Phi_{i_m}-\Phi_{i_n}\| \ .$$

Hence we get that $\{\Phi_{i_k}\}$ is a Cauchy sequence and therefore it converges to a function Φ in the complete space C_o . This convergence is uniform on $[-h,0]$.

Now take the function $v_k(t)$ defined on $[t_o-h,T)$ as follows: $v_k(t) = X_o+\Phi_{i_k}(t-t_o)$, $t\in[t_o-h,t_o]$, $v_k(t) = x(t,t_o,X_o+\Phi_{i_k}) = X_o +$

$$+\int_{t_o}^{t} f(s,x_s(t_o,X_o+\Phi_{i_k}))ds = X_{i_k} - \int_t^T f(s,x_s(t_o,X_o+\Phi_{i_k}))ds \ , \ t\in[t_o,T),$$

$k=1,2,\dots$. We get that $v_k(t)$ converges to $v(t)$: $v(t) = X_o+$ $+ \Phi(t-t_o)$ for $t\in[t_o-h,t_o]$, $v(t) = u(t)$ for $t\in[t_o ,T)$ uniformly on every closed subinterval of $[t_o-h,T)$. We get also that

$$v(t) = X_o + \int_{t_o}^{t} f(s,v_s)ds = Y - \int_t^T f(s,v_s)ds \ , \ t\in[t_o ,T) \ .$$

Thus $v(t) = x(t,t_o,X_o+\Phi)$ and $\lim v(t) = Y$ as $t\to T-$. This proves that $Y\in F(X_o,G)$ and therefore $F(X_o,G)$ is closed.

Finally, we have to prove that $F(X_o,G)$ is connected. Suppose the contrary is true. Then $F(X_o,G)$ can be represented as $F(X_o,G)= =F_1\cup F_2$, where F_i , $i=1,2$, are bounded, closed and disjoint sets. Let $G_i = \{\Phi\in G : F(X_o,\Phi)\in F_i\}$, $i=1,2$. Evidently $G = G_1\cup G_2$ and $G_1\cap G_2 = \emptyset$ and G_1 and also G_2 are nonvoid. Furthermore, the continuous dependence of solutions on the initial functions, Theorem 3 and the closedness of F_i , i=1,2 imply the closedness of G_i , i=1,2 . But then we have that the closed ball G is the union of two sets which are nonvoid, closed and disjoint which is in contradiction with the fact that G is connected.

<u>Remark 1.</u> The constant d in H_3 has to satisfy also the condition $d \leq \int_{t_o}^{t_o+h} \beta(s)ds$ for H_2, H_3 not to contradict each other. In fact, we have

$$d\|\Phi_1-\Phi_2\| \leq \left| \int_{t_o}^{t_o+h} [f(s,z_{1s})-f(s,z_{2s})]ds \right| \leq \int_{t_o}^{t_o+h} \beta(s)\|z_{1s}-z_{2s}\|ds .$$

If we take $z_i(t) = \Phi_i(t-t_o)$, $t\in[t_o-h,t_o]$, $z_i(t) = y(t)$, $t\geq t_o$, i=1,2 , we have that $\|z_{1s}-z_{2s}\| \leq \|\Phi_1-\Phi_2\|$ and from the preceding inequality we get that $d \leq \int_{t_o}^{t_o+h} \beta(s)ds$.

<u>Theorem 6.</u> Let be valid H_1,H_2,H_3 with $\frac{k}{1+k} \leq d \leq \int_{t_o}^{t_o+h} \beta(s)ds$ and H_4:

(H_4) For every two points $X_o,X\in R^n$ and every $y(t)\in B_o$ there is $\Phi\in C_o$ such that for $z(t) = \Phi(t-t_o)$, $t\in[t_o-h,t_o]$, $z(t) = y(t)$, $t\in[t_o ,T)$ the equation

$$X = X_o + \int_{t_o}^{t_o+h} f(t,X_o+z_t)dt$$

holds.

Then the problem (P) has a solution.

Proof. Let $X_1,X_o\in R^n$ be given. Choose $y_1(t)\in B_o$ such that $\lim y_1(t) = X = X_1-X_o$ as $t\to T-$. Then denote

$$Y_1 = X_1 - \int_{t_o+h}^{T} f(t,X_o+y_{1t})dt . \tag{6}$$

With regard to H_4 applied to $X_o,Y_1\in R^n$ and $y_1(t)\in B_o$ there exists $\Phi_1\in C_o$ such that

$$Y_1 = X_o + \int_{t_o}^{t_o+h} f(t,X_o+z_{1t})dt , \tag{7}$$

$z_1(t) = \Phi_1(t-t_o)$ for $t\in[t_o-h,t_o]$ and $z_1(t) = y_1(t)$ for $t\in [t_o, T)$. From (6) and (7) we get

$$(8) \qquad X_1 = X_0 + \int_{t_0}^{T} f(s, X_0 + z_{1s})ds \ .$$

Denote

$$y_2(t) = \int_{t_0}^{t} f(s, X_0 + z_{1s})ds \ , \qquad t \in [t_0 \ ,T) \ .$$

Evidently $y_2(t) \in B_0$ and $\lim y_2(t) = X_1 - X_0 = X$ as $t \to T-$. Now we construct

$$Y_2 = X_1 - \int_{t_0+h}^{T} f(t, X_0 + y_{2t})dt \ .$$

Then with regard to H_4 applied to X_0, Y_2 and $y_2(t)$ there exists $\Phi_2 \in C_0$ such that

$$Y_2 = X_0 + \int_{t_0}^{t_0+h} f(t, X_0 + z_{2t})dt \ ,$$

where $z_2(t) = \Phi_2(t-t_0)$ for $t \in [t_0-h, t_0]$, $z_2(t) = y_2(t)$ for $t \in [t_0 \ ,T)$. Once again we get

$$X_1 = X_0 + \int_{t_0}^{T} f(t, X_0 + z_{2t})dt \ .$$

Put

$$y_3(t) = \int_{t_0}^{t} f(s, X_0 + z_{2s})ds \ , \qquad t \in [t_0 \ ,T) \ .$$

We have that $y_3(t) \in B_0$, $\lim y_3(t) = X_1 - X_0 = X$ as $t \to T-$. Proceeding in this way we get the sequences, $n=2,3,\dots$.

$$(9) \qquad y_n(t) = \int_{t_0}^{t} f(s, X_0 + (z_{n-1})_s)ds \ , \qquad t \in [t_0 \ ,T) \ ,$$

$$(10) \qquad Y_n = X_1 - \int_{t_0+h}^{T} f(t, X_0 + y_{nt})dt \ ,$$

$$(11) \qquad Y_n = X_0 + \int_{t_0}^{t_0+h} f(t, X_0 + (z_n)_t)dt \ ,$$

$z_n(t) = \Phi_n(t-t_0)$ for $t \in [t_0-h, t_0]$, $z_n(t) = y_n(t)$, $t \in [t_0 \ ,T)$

and

$$(12)\qquad X_1 = X_o + \int_{t_o}^{T} f(t,X_o + (z_n)_t)dt\ ,$$

$$(13)\qquad \lim y_n(t) = X_1 - X_o = X \quad \text{as} \quad t \to T- \ .$$

Now from (10), (11) applying H_3 and H_2 we have

$$(14)\qquad \|\Phi_n - \Phi_{n-1}\| \le \Big| \int_{t_o}^{t_o+h} \left[f(t,X_o+z_{nt}) - f(t,X_o+(z_{n-1})_t)\right] dt \Big| \frac{1}{d} =$$

$$= \Big| \int_{t_o+h}^{T} \left[f(t,X_o+y_{nt}) - f(t,X_o + (y_{n-1})_t\right] dt \Big| \frac{1}{d} \le$$

$$\le \frac{1}{d}\int_{t_o+h}^{T} \beta(t) \left\| \left[y_n - y_{n-1}\right]_t \right\| dt \le \frac{1}{d}\int_{t_o+h}^{T} \beta(s)ds \|y_n - y_{n-1}\|_u \le k\|y_n - y_{n-1}\|_u$$

From (9) using H_2 and (14) we get

$$(15)\qquad \|y_{n+1} - y_n\|_u \le \int_{t_o}^{T} \beta(t) \left\| \left[z_n - z_{n-1}\right]_t \right\| dt \le k\|y_n - y_{n-1}\|_u \ .$$

Because $k<1$, (15) means that the sequence $\{y_n(t)\}$ converges uniformly on $[t_o, T)$ to a function $y(t)$. But (14) implies the uniform convergence of the sequence $\{\Phi_n(t)\}$ to a function $\Phi \in C_o$. From all this we conclude that the sequence $\{z_n(t)\}$ converges uniformly on $[t_o-h, T)$ to the function $z(t)$: $z(t) = \Phi(t-t_o)$ for $t \in [t_o-h, t_o]$, $z(t) = y(t)$ for $t \in [t_o, T)$. Then from (9) we get

$$y(t) = \int_{t_o}^{t} f(s,X_o+z_s)ds\ , \qquad t \in [t_o, T)\ .$$

Therefore

$$(16)\qquad X_o + y(t) = X_o + \int_{t_o}^{t} f(s,X_o+z_s)ds\ .$$

Denoting $u(t) = X_o + z(t)$ for $t \in [t_o-h, T)$ we have

$$(17)\qquad u(t) = X_o + \Phi(t-t_o) \quad \text{for} \quad t \in [t_o-h, t_o]\ ,$$

$$u(t) = X_o + \int_{t_o}^{t} f(s,u_s)ds \quad \text{for} \quad t \in [t_o, T)\ .$$

Thus, $u(t)$ is the solution of (1) corresponding to the initial values $(t_o, X_o+\Phi)$. From (12) and (16) we get that $\lim u(t) = X_1$ as $t \to T-$ and $u(t_o) = X_o$. Thus, $u(t)$ is a solution of our problem (P).

<u>Theorem 7.</u> Let H_1, H_2, H_3 be valid and let

$$d > \left[\exp\left(\int_{t_o+h}^{T} \beta(s)ds\right)-1\right] \exp\left(\int_{t_o}^{t_o+h} \beta(s)ds\right) . \tag{18}$$

Then the map $F(X_o, \Phi)$, by fixed X_o , is a one-to-one map of C_o into R^n . This means that in this case the problem (P) has at most one solution.

Proof. Let $\Phi_1, \Phi_2 \in C_o$, $\|\Phi_1 - \Phi_2\| \neq 0$. Then

$$|F(X_o, \Phi_1)-F(X_o, \Phi_2)| = \left|\int_{t_o}^{T} \left[f(t,x_t(t_o,X_o+\Phi_1))-f(t,x_t(t_o,X_o+\Phi_2))\right]\right.$$

$$\left. \cdot dt\right| \geq \left|\int_{t_o}^{t_o+h} \left[f(t,x_t(t_o,X_o+\Phi_1))-f(t,x_t(t_o,X_o+\Phi_2))\right] dt\right| -$$

$$- \left|\int_{t_o+h}^{T} \left[f(t,x_t(t_o,X_o+\Phi_1))-f(t,x_t(t_o,X_o+\Phi_2))\right] dt\right| \geq d\|\Phi_1-\Phi_2\| -$$

$$- \int_{t_o+h}^{T} \beta(s)\|x_s(t_o,X_o+\Phi_1)-x_s(t_o,X_o+\Phi_2)\| ds .$$

Using Lemma 3 from [1] which asserts that, if H_1 and H_2 are valid, the inequality

$$\|x_t(t_o,X_o+\Phi_1)-x_t(t_o,X_o+\Phi_2)\| \leq \|\Phi_1-\Phi_2\| \exp\left(\int_{t_o}^{t} \beta(s)ds\right)$$

holds, we get

$$|F(X_o, \Phi_1)-F(X_o, \Phi_2)| \geq \|\Phi_1-\Phi_2\| \left\{ d + \left[1-\exp\int_{t_o+h}^{T} \beta(s)ds\right] \cdot \exp\int_{t_o}^{t_o+h} \beta(s)ds \right\} \tag{19}$$

which proves our theorem.

<u>Remark 2.</u> If we consider the scalar equation $\dot{x}(t) = a(t)x(t-h)$ where $a(t) \not\equiv 0$ for $t \in [t_o, t_o+h]$, then H_4 will be valid if there is $\Phi \in C_o$ such that $\int_{t_o}^{t_o+h} a(t)\Phi(t-t_o-h)dt \neq 0$. In fact, we have

$$X = X_o + \int_{t_o}^{t_o+h} a(t)(X_o + \lambda\Phi(t-t_o-h))dt = X_o + X_o\int_{t_o}^{t_o+h} a(t)dt + \lambda\int_{t_o}^{t_o+h} a(t)\Phi(t-t_o-h)dt .$$

From this we can calculate λ and then $\lambda\Phi$ will be the sought function.

Remark 3. It can happen that for some given $X_o, X \in R^n$ and $y(t) \in B_o$ there are more than one function $\Phi \in C_o$ satisfying H_4. But if we suppose also the validity of H_3, there can be only one $\Phi \in C_o$ satisfying H_4. In fact, let $\Phi_1, \Phi_2 \in C_o$ be two functions satisfying H_4 for given $X_o, X \in R^n$ and $y(t) \in B_o$. Then we have

$$X = X_o + \int_{t_o}^{t_o+h} f(s, X_o + z_{1s})ds = X_o + \int_{t_o}^{t_o+h} f(s, X_o + z_{2s})ds ,$$

where $z_i(t) = \Phi_i(t-t_o)$ for $t \in [t_o-h, t_o]$, $z_i(t) = y(t)$ for $t \in [t_o, t_o+h]$, $i=1,2$. Applying H_3 we get

$$0 = \left| \int_{t_o}^{t_o+h} [f(s, X_o + z_{1s}) - f(s, X_o + z_{2s})]ds \right| \geq d\|\Phi_1 - \Phi_2\| .$$

Thus $\Phi_1 = \Phi_2$.

It would be desirable to clear up the relation between H_3 and H_4. It seems to us that both hypotheses H_3 and H_4 can be substituted by another one from which both H_3 and H_4 follow. This problem will be discussed in another paper.

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