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NON LINEAR QUASI VARIATIONAL INEQUALITIES AND STOCHASTIC IMPULSE CONTROL THEORY

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1. It has been shown by A. Bensoussan and J.L.Lions [1] [2] that the Hamilton-Jacobi function related to a stochastic optimal control problem with both continuous and impulse control can be obtained as the strong regular solution of a quasi-variational inequality involving a second order semi-linear partial differential operator.

The aim of this paper is to prove the existence of such a regular solution, for the stationary case and Dirichlet boundary condition, under the assumption that the higher order coefficients of the partial differential operator are constants. We shall also prove that the solution, which is unique, is the limit of monotone iterative algorithms converging from above and from below.

2. The inequality (Q.V.I.) we are interested in can be stated as follows:

$$(1)\quad \begin{cases} u \in H^1_0(\mathcal{O}) \cap C(\bar{\mathcal{O}}), \quad Lu \in L^2(\mathcal{O}) \\ u(x) \leq M(u)(x) \quad \text{for all } x \in \bar{\mathcal{O}} \\ Lu + G(u) \leq f \quad \text{a.e. in } \mathcal{O} \\ (u-M(u))(Lu+G(u)-f) = 0 \quad \text{a.e. in } \mathcal{O} \end{cases}$$

where M(u) is defined by

$$(2)\qquad M(u)(x) = 1 + \inf_{\substack{\xi \in \mathbb{R}^N_+ \\ \xi \geq 0}} u(x+\xi)$$

and the data of the problem are:

(i) $\mathcal{O}$, a bounded open subset of $\mathbb{R}^N$, with a smooth boundary Γ, say Γ of class C^2;

(ii) L, a second order linear uniformly elliptic partial differential operator in divergence form,

$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{j=1}^{N} a_j \frac{\partial u}{\partial x_j} + a_o u ,$$

whose coefficients satisfy the conditions

$$a_{ij} \in C^1(\bar{\mathcal{O}}) \;,\; a_j, a_o \in L^\infty(\mathcal{O}) \;,\; a_o \geq 0 \text{ a.e., } \forall i,j = 1,\dots,N \;;$$

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq \beta \sum_{i=1}^{N} \xi_i^2 \quad \text{a.e. in } \mathcal{O}, \text{ for all } \xi \in \mathbb{R}^N ,$$

where $\beta > 0$ is some given constant;

(iii) G, a first order nonlinear partial differential operator of the form

$$G(u)(x) = -H(x, Du(x))$$

where $Du = \left\{\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right\}$, the (real valued) function

$$H(x, p) \quad , \quad x \in \mathcal{O} \;,\; p \in \mathbb{R}^N$$

being concave in p for a.e. $x \in \mathcal{O}$ and satisfying in addition the conditions:

$$|H(x,p)| \leq h(x) + c_o|p| \qquad \text{a.e. } x \in \mathcal{O} \;,\; \forall p \in \mathbb{R}^N$$

$$|H(x,p') - H(x,p'')| \leq c_o|p'-p''| \quad \text{a.e. } x \in \mathcal{O}, \forall p',p'' \in \mathbb{R}^N$$

for some constant $c_o \geq 0$ and some $h \in L^\infty(\mathcal{O})$;

(iv) f, a given function of $L^\infty(\mathcal{O})$.

We also assume that the following conditions are verified:

(v) $$\inf_{\mathcal{O}} a_o \geq (2\beta)^{-1}\left\{\sum_{j=1}^{N} \|a_j\|_\infty^2 + 2c_o^2\right\}$$

(vi) $$\inf_{\mathcal{O}} f - \sup_{\mathcal{O}} h \geq \inf_{\mathcal{O}} a_o .$$

We shall come back to the last assumption (vi) in the following Remark 2 and Remark 3.

Our first result is the following (see U.Mosco [3]).

THEOREM 1. Let us suppose, in addition to what required in (i) ...

(vi) above, that the coefficients a_{ij} of the operator L are constants in $\mathcal{O}$. Then, the solution u of problem (1) exists and is unique. Moreover, it satisfies the additional regularity properties

$$(3) \qquad u \in W^{2,p}(\mathcal{O}, \quad \forall p \geq 2 \quad , \quad Lu \in L^{\infty}(\mathcal{O}),$$

in particular,

$$u \in C^{1,\alpha}(\bar{\mathcal{O}}) \quad \text{for all} \quad 0 < \alpha < 1. \quad \blacksquare$$

REMARK 1. The specific function H that appears in the QVI arising from the stochastic impulse control theory is the so called *Hamiltonian function*, defined by

$$(4) \qquad H(x,p) = \min_{d \in U} \left[g_0(x,d) + p \cdot g_1(x,d) \right] ,$$

where U is a subset of some $\mathbb{R}^Q$, $Q \geq 1$, $g_0 : \bar{\mathcal{O}} \times U \to \mathbb{R}$ and $g_1 : \bar{\mathcal{O}} \times U \to \mathbb{R}^N$. If, for instance, we assume that U is compact and that g_0 and g_1 are continuous in $d \in U$ for fixed x a.e. $\in \mathcal{O}$ and continuous in $x \in \bar{\mathcal{O}}$ uniformly with respect to $d \in U$, then the Hamiltonian (4) verifies the properties required in (iii) above, with $c_0 \geq 0$ any constant and $h \in L^{\infty}(\mathcal{O})$ any function such that

$$|g_0(x,d)| \leq h(x) \qquad \forall x \text{ a.e.} \in \mathcal{O} \qquad \forall d \in U ,$$

$$|g_1(x,d)| \leq c_0 \qquad \forall x \in \bar{\mathcal{O}} , \quad \forall d \in U ;$$

see also A.Bensoussan, J.L.Lions, *loc. cit.* $\blacksquare$

3. Before stating further results, let us introduce the notion of *weak solution* of the QVI considered above.

We introduce the bilinear form

$$(5) \quad a(v,w) = \sum_{i,j=1}^{N} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} dx + \sum_{j=1}^{N} \int_{\mathcal{O}} a_j \frac{\partial v}{\partial x_j} w \, dx + \int_{\mathcal{O}} a_0 vw dx$$

which is well defined for any functions v,w in the Sobolev space $H^1(\mathcal{O})$, and we denote by $(\cdot,\cdot)$ the $L^2(\mathcal{O})$ inner product. If $\langle \cdot,\cdot \rangle$ denotes the duality pairing between the space $H^1_0(\mathcal{O})$ and its dual $H^{-1}(\mathcal{O})$, the identity

(6) $\langle Lu, w \rangle = a(u,w)$,

when $u \in H^1(\mathcal{O})$, $w \in H^1_o(\mathcal{O})$, defines L as an operator from the space $H^1(\mathcal{O})$ to the dual $H^{-1}(\mathcal{O})$ of $H^1_o(\mathcal{O})$, and the identity

(7) $\langle A(u),w \rangle = a(u,w) + (G(u),w)$

also defines the non linear operator

(8) $A = L + G$

from the space $H^1(\mathcal{O})$ to $H^{-1}(\mathcal{O})$.
For any given function

(9) $\psi : \mathcal{O} \to \mathbb{R}$ measurable, with $\psi \geq 0$ a.e. on Γ

we denote by $\sigma^A_f(\psi)$ the (unique) solution v of the following *variational inequality* (V.I.)

$$
(10) \quad \begin{cases} v \in H^1_o(\mathcal{O}), \quad v \leq \psi \quad \text{a.e. in } \mathcal{O} \\ \langle A(v), v-w \rangle \leq (f,v-w) \\ \forall w \in H^1_o(\mathcal{O}), \ w \leq \psi \quad \text{a.e. in } \mathcal{O} \end{cases}
$$

where A is the operator (8).

For any function

(11) $u \in L^\infty(\mathcal{O}$, with $u \geq -1$ a.e. in $\mathcal{O}$,

the function $\psi = M(u)$ - with $M(u)$ defined by (2), where the inf is now taken as the ess inf in the space $L^\infty(\mathcal{O})$ - clearly verifies assumption (9)
Therefore, we can consider the function

(12) $(\sigma^A_f \circ M)(u) = \sigma^A_f(M(u))$:

according to the definition of σ^A_f just given, the function (12) is the solution of the following V.I.

$$(13)\quad \begin{cases} v \in H^1_o(\mathcal{O}), & v \leq M(u) \text{ a.e. in } \mathcal{O} \\ \langle A(v), v-w \rangle \leq (v, v-w) & \\ \forall w \in H^1_o(\mathcal{O}), & w \leq M(u) . \end{cases}$$

We say that a function u defined in $\mathcal{O}$ is a (weak) *subsolution* of problem (1) if u satisfies the condition (11) and, moreover,

$$(14)\qquad u \leq (\sigma^A_f \circ M)(u) \qquad \text{a.e. in } \mathcal{O}.$$

The function u is said a *supersolution* of (1) if in addition to (11) we have

$$(15)\qquad u \geq (\sigma^A_f \circ M)(u) \qquad \text{a.e. in } \mathcal{O}.$$

We say that u is a *weak solution* of problrm (1) if u verifies (11) and moreover

$$(16)\qquad u = (\sigma^A_f \circ M)(u) \qquad \text{a.e. in } \mathcal{O},$$

that is, u is a *fixed point* of the mapping $\sigma^A_f \circ M$. Let us remark, incidentally, that in corresponce of the L^∞-estimates for problem (13), the map $\sigma^A_f \circ M$ carries the subset of $L^\infty(\mathcal{O})$ defined by (11) into a subset of $H^1_o(\mathcal{O}) \cap L^\infty(\mathcal{O})$.

According to the definition above, a weak solution of problem (1) is thus any function u that satisfies the conditions

$$(17)\quad \begin{cases} u \in H^1_o(\mathcal{O}) \cap L^\infty(\mathcal{O}), \quad u \geq -1 \quad \text{a.e. in } \mathcal{O} \\ u \leq M(u) \quad \text{a.e. in } \mathcal{O} \\ a(u,u-w) + (\ (u), u-w) \leq (f, u-w) \\ \forall w \in H^1_o(\mathcal{O}), \ w \leq M(u) \text{ a.e. in } \mathcal{O} \end{cases}$$

where a(u,v) is the form (5). It is quite easy to check that if u is the solution of (1) (and we then refer to u as to the *strong* solution of problem (1)), then u is in particular a weak solution in the sense just defined.

REMARK 2. Under the assumptions of Theorem 1, in particular as a consequence of (vi), the function $z \equiv -1$ in O is a subsolution of problem (1). It follows, in fact, from the L^∞ estimates already mentioned, that if $\psi = 1$ and (vi) holds, then the solution $v=\sigma_f^A(1)$ of (10) satisfies the condition $-1 \leq v$ a.e. in O, which is to say, due to the fact that $z \equiv -1$ implies $M(z) \equiv 1$, $z \leq (\sigma_f^A \circ M)(z)$ a.e. in O. ■

REMARK 3. In Theorem 1 it suffices to assume, in place of (vi), the weaker hypothesis

(vii) It exists a subsolution z of problem (1) ($z \geq -1$ a.e. in O).

The hypothesis is satisfied provided $f - h$ is not too negative in O, which is indeed also a *necessary* condition for the existence of the solution u of (1), as it can be easily checked directly on (1) taking into account the comparison theorems and the fact that u vanishes on Γ. ■

THEOREM 2. Let us suppose that the hypothesis of Theorem 1 are satisfied, with (vi) possibly replaced by the weaker assumption (vii). Let z be a function verifying (vii) and let u^o be any given function such that

$$u^o \in H_0^1(O), \quad A(u^o) \in L^\infty(O), \quad u^o \geq z \text{ a.e. in } O. \tag{18}$$

Then, the solution u^n of the V.I.

$$u^n = (\sigma_f^A \circ M)(u^{n-1}) \tag{19}$$

defined iteratively for $n = 1,2,\ldots$, exists for every n and converges as $n \to \infty$ to the (strong) solution u of problem (1) in the weak topology of $W^{2,p}(O)$ for any $p \geq 2$, the sequence $\{A(u^n)\}$ being bounded in $L^\infty(O)$. Furthermore, if u^o is a supersolution [subsolution, resp.] of problem (1), then the sequence $\{u^n\}$ is non-increasing [non-decreasing, resp.] in O. ■

Theorem 1 is clearly a consequence of Theorem 2, so we shall prove Theorem 2 in what follows.

Let us remark that when $G = 0$, that is, $H \equiv 0$, and *all* coefficients of L are constants, then the existence results stated above were proved by J.L.JOLY, U.MOSCO and G.M.TROIANIELLO [1] [2], who also obtained a dual pointwise estimate of the solution.

The existence of a *weak* solution of problem (1) has been proved by A.BENSOUSSAN and J.LIONS, *loc.cit.*, by using the theory of *monotone operators*.

Under the assumptions (i)-(vi) above, the weak solution of (1) is also unique. This can be proved by extending to the case at hand the uniqueness result due to T.LAETSCH [1] for the linear case A = L and $f \geq 0$, in a modified form communicated to the author by J.L.JOLY, allowing arbitrary $f \in L^\infty(\mathcal{O})$. It should be also remarked that the uniqueness of the *strong* solution of problem (1), when H is given by (4), also follows *via* the interpretation of u as the Hamilton-Jacobi function of the related control problem, as shown in A.BENSOUSSAN and J.L.LIONS, *loc.cit.*.

When all coefficients of L are constants and the function H(x,p) is of the form $H(x,p) = H_0(x) + H_1(p)$, then a pointwise dual estimate of A(u) a.e. in $\mathcal{O}$, can be also obtained, see U.MOSCO [2].

4. Let $\xi \in \mathbb{R}^N_+$. We define the map

$$(20) \qquad \pi_\xi \circ \tau_\xi : H^1_0(\mathcal{O}) \cap L^\infty(\mathcal{O}) \to H^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$$

by setting

$$\begin{cases} \pi_\xi \circ \tau_\xi(v)(x) = \tau_{-\xi}(v)(x) = v(x+\xi) & \text{if } x \in \mathcal{O}'_\xi \\ \pi_\xi \circ \tau_\xi(v)(x) = 0 & \text{if } x \in \mathcal{O}-\mathcal{O}'_\xi \end{cases}$$

where $\mathcal{O}'_\xi$ is the, possibly empty, open subset of all $\xi \in \mathcal{O}$ such that $x + \xi \in \mathcal{O}$.

We define the map

$$(21) \qquad \pi'_\xi \circ A'_\xi \circ \tau'_\xi : H^1_0(\mathcal{O}) \cap H^2(\mathcal{O}) \to L^2(\mathcal{O}) \ ,$$

by setting

$$\begin{cases} \pi'_\xi \circ A'_\xi \circ \tau'_\xi(v)(x) = A'_\xi(\tau'_{-\xi}(v))(x) & \text{if } x \in \mathcal{O}'_\xi \\ \pi'_\xi \circ A'_\xi \circ \tau'_\xi(v)(x) = A(0)(x) = -H(x,0) & \text{if } x \in \mathcal{O}-\mathcal{O}'_\xi \ , \end{cases}$$

where A'_ξ denotes the restriction of the operator A to the open set $\mathcal{O}'_\xi$ and $\tau'_{-\xi}(v)$ the restriction to $\mathcal{O}'_\xi$ of the translated function $\tau_{-\xi}(v)(x) = v(x + \xi)$. We have indeed

$$(22) \qquad \pi'_\xi \circ A'_\xi \circ \tau'_\xi : H^1_0(\mathcal{O}) \cap W^{2,p}(\mathcal{O}) \to L^p(\mathcal{O})$$

for every $p \geq 2$.

Let $\mathcal{F}$ denote the family of all *finite* subsets F of $\mathbb{R}^N_+$, such that $0 \in F$. For each fixed $F \in \mathcal{F}$, we define the map

$$(23) \qquad M_F : H^1_o(\mathcal{O}) \cap L^\infty(\mathcal{O}) \to H^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$$

by setting

$$M_F(v) = 1 + \inf_{\xi \in F} \pi_\xi \circ \tau_\xi(v) \wedge 0 \ ,$$

where both *inf* and $\wedge$ denote the usual a.e. lattice operation in $L^\infty(\mathcal{O})$.

The main point in the proof of Theorem 2 consists in proving the uniform dual estimate stated in the following

PROPOSITION 1. For every fixed $F \in \mathcal{F}$ let us consider the sequence $\{u^n_F\}$ defined recursively by

$$(24) \qquad u^o_F = u^o \quad , \qquad u^n_F = (\sigma^A_f \circ M_F)(u^{n-1}_F)$$

where $(\sigma^A_f \circ M_F)(u^{n-1}_F) = \sigma^A_f(M_F(u^{n-1}_F))$ is obtained for each n=1,2,... as the solution of the V.I. (10) for $\psi = M_F(u^{n-1}_F)$. Then all solutions u^n_F's exist and they verify the following dual estimate

$$(25) \qquad \|A(u^n_F)\|_{L^\infty(\mathcal{O})} \leq c$$

with a constant c which is *independent on* $n \geq 1$ *and* $F \in \mathcal{F}$. ∎

The proof of the Proposition is done in four steps, the first two of which rely only on *potential* theoretic properties of the nonlinear operator A.

The first step consists in proving that for any function $v \in H^1_o(\mathcal{O})$, such that $A(v) \in L^p(\mathcal{O})$ with $p \geq 2$, the distribution $A(M_F(v))$ of $H^{-1}(\mathcal{O})$ is actually a *measure* that can be estimated from below, in the sense of measures, by a L^p function (depending on F), namely

$$(26) \quad A(M_F(v)) \geq \inf_{\xi \in F} \pi'_\xi \circ A'_\xi \circ \tau'_\xi(v) \wedge A(0) .$$

For more details we refer to U.MOSCO [2].

The second step relies on the *dual estimates* for nonlinear V.I., see Th. 4.1 in U.Mosco [1].By assuming that $\psi \in H^1(\mathcal{O})$ and $A(\psi)$ is a measure in $\mathcal{O}$, the solution v of the V.I. (10) is shown to satisfy in

$\mathcal{O}$ the estimate

(27) $$f \geq A(v) \geq f \wedge A(\psi)$$

in the sense of measures.

We now consider the sequence of iterates (24) for a given $F \in \mathcal{F}$. Since the initial function u^o is assumed to satisfy (18), we may suppose that u_F^{n-1} verifies

(28) $$u_F^{n-1} \in H_o^1(\mathcal{O}) \quad , \quad A(u_F^{n-1}) \in L^p(\mathcal{O}), \text{ with } p \geq 2.$$

As in the first step above, it follows that $A(M_F(u_F^{n-1}))$ is a measure, and the estimate (26) holds, with $v = u_F^{n-1}$. According to the second step above, the estimate (27) also holds, with $v = u_F^{n-1}$. It follows then that u_F^n satisfies the inequalities

(29) $$f \geq A(u_F^n) \geq f \wedge_{\xi \in F} \pi'_\xi \cdot A'_\xi \circ \tau'_\xi(u_F^{n-1}) \wedge A(0), \text{ a.e. in } \mathcal{O}.$$

By well known regularity theorems, it also follows from (28) that

$$u_F^{n-1} \in W^{2,p}(\mathcal{O}) ,$$

hence, by (22),

$$\pi'_\xi \circ A'_\xi \circ \tau'_\xi(u_F^{n-1}) \in L^p(\mathcal{O})$$

for every $\xi \in \mathbb{R}_+^N$. Since $f \in L^p(\mathcal{O})$ and $A(0) = -H(x,0) \in L^p(\mathcal{O})$ by our assumptions (iii) and (iv), it follows then from (29) that u_F^n too verifies the properties (28). Thus the estimate (29) holds *for all* $n \geq 1$.

Let us also remark that the following uniform estimate holds

(30) $$\|u_F^n\|_{L^\infty(\mathcal{O})} \leq 1 \qquad \forall n \geq 1 , \forall F \in \mathcal{F}.$$

In fact, since $u^o \geq z \geq -1$, we may assume, for given F, that $u_F^{n-1} \geq z \geq -1$, hence

$$1 \geq M_F(u_F^{n-1}) \geq M_F(z)$$

It follows, by well know comparison theorems, that

$$\sigma_f^A(M_F(u_F^{n-1})) \geq \sigma_f^A(M_F(z)) ,$$

therefore

$$u_F^n \geq z$$

since z is a subsolution; on the other hand,

$$1 \geq M_F(u_F^n) \geq u_F^n .$$

Thus

$$1 \geq u_F^n \geq z \geq -1 \quad \text{a.e. in } \mathcal{O} ,$$

for all $n \geq 1$.

Let us note incidentally that the last property assures that the sequence of iterates $\{u_F^n\}_{n\geq 1}$ actually exists, each u_F^n being the solution of the V.I. (13), where $M(u)$ is replaced by $M_F(u_F^{n-1})$; in fact, $M_F(u_F^{n-1}) \geq 0$ a.e. in $\mathcal{O}$.

Up to now no use has been made of the assumption that the coefficients a_{ij}'s of the operator L are constants. This property of L, however, will now be used in the following argument.

Let us denote by L_o the leading part of the operator L, i.e.

$$L_o v = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij} \frac{\partial v}{\partial x_j}) ,$$

and for any $\xi \in \mathbb{R}_+^N$ such that $\mathcal{O}_\xi' \neq \emptyset$, according to our previous notation for the operator A_ξ', let us also denote by $(L_o)_\xi'$ the restriction of the operator L_o to the open set $\mathcal{O}_\xi'$ and by $\pi_\xi' \circ (L_o)_\xi' \cdot \tau_\xi'$ the operator (21), with $(L_o)_\xi'$ playing the role of A_ξ'.

Since L_o commutes with the translations, we have

$$\| (L_o)_\xi' (\tau_{-\xi}'(v)) \|_{L^\infty(\mathcal{O}_\xi')} \leq \| L_o(v) \|_{L^\infty(\mathcal{O})}$$

for all $\xi \in \mathbb{R}_+^N$ such that $\mathcal{O}_\xi' \neq \emptyset$. Hence also, by our assumption (iii) on the function H,

$$\text{(31)} \quad \| \pi_\xi' \circ (L_o)_\xi' \circ \tau_\xi'(v) \|_{L^\infty(\mathcal{O})} \leq \| L_o(v) \|_{L^\infty(\mathcal{O})} , \quad \forall \xi \in \mathbb{R}_+^N .$$

Let now v be any function satisfying

(32) $v \in H^1_o(\mathcal{O})$, $A(v) \in L^\infty(\mathcal{O})$.

We then have

(33) v , Dv , $L_o v \in L^\infty(\mathcal{O})$

and by simple interpolation estimates it also follows from (31) and the hypotheses (ii) and (iii) that

(34) $\pi'_\xi \circ A'_\xi \circ \tau'_\xi(v) \in L^\infty(\mathcal{O}) \qquad \forall \xi \in \mathbb{R}^N_+$

and

(35) $\|\pi'_\xi \circ A'_\xi \circ \tau'_\xi(v)\|_{L^\infty(\mathcal{O})} \leq c$

with $c > 0$ a constant independent on ξ.

We are now in a position to make the third step of the proof, which consists in proving that

(36) $A(u^n_F) \in L^\infty(\mathcal{O})$, $\forall n \geq 0$, $\forall F \in \mathcal{F}$.

Since $A(u^o_F) = A(u^o) \in L^\infty(\mathcal{O})$ by (18), in order to prove (36) it suffices to show that for every $F \in \mathcal{F}$ and $n \geq 1$,

(37) $A(u^{n-1}_F) \in L^\infty(\mathcal{O})$

implies

(38) $A(u^n_F) \in L^\infty(\mathcal{O})$.

In fact, by (37) the function $v = u^{n-1}_F$ satisfies the hypothesis (32), hence (34) holds, that is,

$$\pi'_\xi \circ A'_\xi \circ \tau'_\xi(u^{n-1}_F) \in L^\infty(\mathcal{O}) \ ;$$

moreover, by the assumptions (iii) and (iv), we also have $A(0) = -H(x,0) \in L^\infty(\mathcal{O})$ and $f \in L^\infty(\mathcal{O})$. Therefore (38) is an immediate consequence of the recursive estimate (29).

The last step of the proof of Proposition 1 consists in proving that the uniform estimate (25) holds. Again by trivial interpolation estimates, it suffices prove that

(39) $$\|L_o u_F^n\|_{L^\infty(\mathcal{O})} \leq c \qquad \forall n \geq 0 \quad \forall F \in \mathcal{F},$$

with c some constant independent on n and F.

Let us remark first that for every $F \in \mathcal{F}$ each iterate u_F^n, $n \geq 1$, can be regarded - with notation taken from above - as the solution of the VI

(40) $$u_F^n = \sigma_{f_{n,F}}^{L_o}(\bar{M}_F(u_F^{n-1}))$$

where

(41) $$f_{n,F} = f - A_1(u_F^n)$$

and

$$A_1(u_F^n) = A(u_F^n) - L_o(u_F^n) = \sum_{j=1}^{n} a_j \frac{\partial u_F^n}{\partial x_j} + a_o u_F^n + G(u_F^n).$$

By the assumptions (ii) (iii) and (iv), we have

$$\|f_{n,F}\|_{L^\infty(\mathcal{O})} \leq c\|Du_F^n\|_{L^\infty(\mathcal{O})} + c\|u_F^n\|_{L^\infty(\mathcal{O})} + c$$

for every $n \geq 1$ and $F \in \mathcal{F}$, hence also, by (30),

(42) $$\|f_{n,F}\|_{L^\infty(\mathcal{O})} \leq c\|Du_F^n\|_{L^\infty(\mathcal{O})} + c$$

for some constants c independent on n and F.

As a consequence of (36), we have

(43) $$L_o(u_F^n) \in L^\infty(\mathcal{O})$$

moreover, $Du_F^n \in L^\infty(\mathcal{O})$, hence also, from (42)

(44) $$f_{n,F} \in L^\infty(\mathcal{O}) \quad , \quad \forall n \geq 1 \quad \forall F \in \mathcal{F}.$$

We are thus in a position, for every $F \in \mathcal{F}$ and each fixed $n \geq 1$, to make use of estimate (29), with

$$A = L_o \qquad \text{and} \qquad f = f_{n,F} .$$

We thus have

(45) $$f_{n,F} \geq L_o u_F^n \geq f_{n,F} \bigwedge_{\xi\in F} \pi'_\xi \circ (L_o)'_\xi \circ \tau'_\xi (u_F^{n-1}) \wedge 0 \quad \text{a.e. in } \mathcal{O},$$

for every $F \in \mathcal{F}$ and all $n \geq 1$.

By taking (31) and (44) into account, it follows from (45) that

(46) $$\|L_o u_F^n\|_{L^\infty(\mathcal{O})} \leq \max\left\{\|f_{n,F}\|_{L^\infty(\mathcal{O})}, \|L_o(u_F^{n-1})\|_{L^\infty(\mathcal{O})}\right\}$$

for every $F \in \mathcal{F}$ and all $n \geq 1$.

On the other hand, by (42),(30) and classical interpolation estimates, we have

(47) $$\|f_{n,F}\|_{L^\infty(\mathcal{O})} \leq c\|L_o u_F^n\|^\delta_{L^\infty(\mathcal{O})} + c \ ,$$

with $\frac{1}{2} < \delta < 1$ and c some constants independent on $n \geq 1$ and $F \in \mathcal{F}$.

By taking (43) into account, the uniform estimate (39) is then an immediate consequence of (46) and (47).

The proof of Proposition 1 has thus been completed.

5. Let us now consider the sequence of iterates (19). The existence of u^n, for every $n \geq 1$, is shown as before the existence of the iterates u_F^n.

We then prove that the sequence $\{u^n\}_{n\geq 1}$ verifies the uniform estimate

(48) $$\|A(u^n)\|_{L^\infty(\mathcal{O})} \leq c$$

with a constant c independent on n.

In fact, let for each $k=1,2,\ldots$, $F_k \in \mathcal{F}$ be such that

(49) $$d_{F_k,\mathcal{O}} \leq \frac{1}{k}$$

where for each $F \in \mathcal{F}$ we define

$$d_{F,\mathcal{O}} = \sup_{\substack{\xi \in \mathbb{R}^N_+ \\ |\xi| \leq \operatorname{diam} \mathcal{O}}} \operatorname{dist}(\xi, F)$$

In consequence of the estimate (25), (a subsequence of) the $\{u^n_{F_k}\}_{k\geq 1}$ converges to a function

$$\tilde{u}^n \in H^1_0(\mathcal{O}) \cap W^{2,p}(\mathcal{O}) \quad , \quad p \geq 2 \ ,$$

weakly in $W^{2,p}(\mathcal{O})$ for all $p \geq 2$ and strongly in $H^1_o(\mathcal{O})$. Therefore, again by (25), the sequence $\{A(u^n_{F_k})\}_{k\geq 1}$ converges to $A(\tilde{u}^n)$ in the weak* topology of $L^\infty(\mathcal{O})$ and

(50) $$\|A(\tilde{u}^n)\|_{L^\infty(\mathcal{O})} \leq c,$$

with c a constant independent on n.

We then prove that for every $F \in \mathcal{F}$ and every $n \geq 1$, we have

(51) $$\|u^n - u^n_F\|_{L^\infty(\mathcal{O})} \leq c\, n\, d_{F,\mathcal{O}}$$

with c independent on $n \geq 1$ and $F \in \mathcal{F}$, hence $\tilde{u}^n = u^n$ for all $n \geq 1$ and (48) follows from (50). For more details on (51) we refer to U.Mosco [2].

Let us now consider the sequence $\{u^n\}$, beginning with the case when u^o is a super-solution or a sub-solution. Then, as a consequence of comparison theorems, $\{u^n\}$ is monotone; by taking (48) into account, we can conclude that the sequence $\{u^n\}$ converges to some function $\tilde{u}$ weakly in $W^{2,p}(\mathcal{O})$ and strongly in $H^1_o(\Omega)$ and $L^\infty(\mathcal{O})$. On the other hand, since $\tilde{u} \geq -1$ in $\mathcal{O}$, the solution $\sigma^A_f \circ M(\tilde{u})$ exists and we have for every $n \geq 1$

$$\|u^n - \sigma^A_f \circ M(\tilde{u})\|_{L^\infty(\mathcal{O})} = \|\sigma^A_f \circ M(u^{n-1}) - \sigma^A_f \circ M(\tilde{u})\|_{L^\infty(\mathcal{O})} \leq \|u^{n-1} - \tilde{u}\|_{L^\infty(\mathcal{O})},$$

since the map $\sigma^A_f \circ M$ is non-expansive for the L^∞ norm. Therefore, u^n converges to $\sigma^A_f \circ M(\tilde{u})$ in $L^\infty(\mathcal{O})$ as $n \to \infty$, hence

$$\tilde{u} = \sigma^A_f \circ M(\tilde{u}) \qquad \text{a.e. in } \mathcal{O}$$

which is to say, the function (52) coincides a.e. in $\mathcal{O}$ with the (unique) solution u of problem (1).

As for $\{u^n\}$ constructed from u^o as in the general case, let us remark that, by comparison theorems, u^n verifies

$$(\sigma^A_f \circ M)^n z \leq u^n \leq (\sigma^A_f \circ M)^{n-1}\bar{u},$$

and that, from what we have already proved, both sequences $\{(\sigma^A_f \circ M)^n z\}$ and $\{(\sigma^A_f \circ M)^n \bar{u}\}$ converge to u in $L^\infty(\Omega)$. This suffices, by taking (48) again into account, to complete the proof of the theorem.

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