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THE DIRICHLET PROBLEM

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Given a partial differential operator L of second order on a relatively compact open subset V of $\mathbb{R}^n$ and a continuous real function f on V^* the corresponding Dirichlet problem consists in finding a continuous real function u on $\overline{V}$ such that $Lu = 0$ on V and $u = f$ on V^*.

Since about twenty years ([1], [4])it is well known that a general treatment of this question is possible by using the concept of a harmonic space. We shall sketch how this is done and then discuss some recent developments.

1. Harmonic spaces

Let X be a locally compact space with countable base. For every open U in X let $\mathcal{H}(U)$ be a linear space of continuous real functions on U, called harmonic functions on U, and suppose that $\mathcal{H} = \{\mathcal{H}(U) : U$ open in $X\}$ is a sheaf.

Standard examples. 1. Laplace equation. X relatively compact open $\subset \mathbb{R}^n$, $\mathcal{H}(U) = \{u \in C^2(U) : \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0\}$ 2. Heat equation. X relatively compact open $\subset \mathbb{R}^{n+1}$, $\mathcal{H}(U) = \{u \in C^2(U) : \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial u}{\partial x_{n+1}}\}$.

A relatively compact open subset V of X is called regular if for every $f \in C(V^*)$ there exists a unique extension $H^V f$ on $\overline{V}$ which is harmonic on V and positive if f is positive.

Let us suppose that $(X,\mathcal{H})$ has the following properties:

I. The regular sets form a base of X.

II. For every open U in X and increasing sequence (h_n) of harmonic functions on U such that $h := \sup h_n$ is locally bounded the function h is harmonic on U.

III. $1 \in \mathcal{H}(X)$, $\mathcal{H}^+(X)$ separates the points of X.

Then $(X,\mathcal{H})$ is a harmonic space.

Remark. We note that the general concept of a harmonic space in the sense of Constantinescu-Cornea [4] uses a slightly weaker form of property (I) and a separation property which is considerably weaker than our property (III). Accepting some technical modifications all the material we want to discuss can be presented in the more general situation (see [2], [3]). But probably the essential ideas become more clear in our setup.

Let V be a regular set and $x \in V$. Then the mapping $f \longmapsto H^V f(x)$ is a positive linear form on $C(V^*)$, hence a positive Radon measure μ_x^V on V^*, called the <u>harmonic measure</u> (on V at x).

2. The Dirichlet problem and the PWB-method

Let U be a relatively compact open subset of X. Given a function $f \in C(U^*)$ the corresponding Dirichlet problem asks for a continuous extension of f to a function $h \in C(\overline{U})$ which is harmonic in U. Therefore, one is interested in the linear space

$$H(U) := \{h \in C(\overline{U}) : h \text{ harmonic in } U\} .$$

If this Dirichlet problem is solvable for every $f \in C(U^*)$ then U is regular, $H(U) \cong C(U^*)$, and vice versa. However, U may be not regular and then there are functions $f \in C(U^*)$ for which the Dirichlet problem is not solvable.

But there is a method due to Perron, Wiener and Brelot (PWB-method) which yields a positive linear mapping $f \longmapsto H^U f$ such that $H^U f$ is harmonic on U for every $f \in C(U^*)$ and such that $H^U f$ is the solution of the Dirichlet problem provided a solution exists.

The PWB-method of determining a so-called <u>generalized solution</u> of the Dirichlet problem uses hyperharmonic functions. A l.s.c. function $v : U \to]-\infty, +\infty]$ is called <u>hyperharmonic</u> (on U) if $\mu_x^V(v) \le v(x)$ for every regular V such that $\overline{V} \subset U$ and for every $x \in V$.

Let ${}^*H(U) = \{v | v : \overline{U} \to]-\infty, +\infty] \text{ l.s.c., } v \text{ hyperharmonic on } U\}$. We note that ${}^*H(U) \cap -{}^*H(U) = H(U)$. ${}^*H(U)$ is a convex cone satisfying the following boundary minimum principle: If $v \in {}^*H(U)$ and $v \ge 0$ on U^* then $u \ge 0$ on $\overline{U}$.

Let $f \in C(U^*)$. Defining

$$\overline{H}^U f = \inf \{v \in {}^*H(U) : v \ge f \text{ on } U^*\} ,$$

$$\underline{H}^U f = \sup \{w \in -{}^*H(U) : w \le f \text{ on } U^*\}$$

the boundary minimum principle yields $\underline{H}^U f \le \overline{H}^U f$. If the Dirichlet problem for f is solvable, i.e. if there exists a function $h \in H(U)$ such that $h = f$ on U^* then evidently $h \le \underline{H}^U f$ and $\overline{H}^U f \le h$, hence $\underline{H}^U f = \overline{H}^U f = h$.

It can be shown that for every $f \in C(U^*)$

$$\overline{H}^U f = \underline{H}^U f =: H^U f$$

and furthermore $H^U f$ is harmonic on U, $H^U f = f$ on U^*.

A boundary point $z \in U^*$ is called <u>regular</u> if for all $f \in C(U^*)$ the generalized solution $H^U f$ is continuous at z. Evidently, U is regular if and only if all boundary points of U are regular. The generalized solution of the Dirichlet problem and a useful criterion for the regularity of boundary points can be obtained using balayage of measures.

3. Balayage

Let ${}^*H^+$ denote the set of all positive hyperharmonic functions on X. Given an arbitrary subset A of X and a function $u \in {}^*H^+$ one tries to find a smallest function $v \in {}^*H^+$ satisfying $v = u$ on A. The obvious candidate is the <u>pre-sweep</u> (or rēduite function)

$$R_u^A := \inf \{v \in {}^*H^+ : v = u \text{ on } A\} .$$

Since R_u^A is not l.s.c. in general, one replaces R_u^A by the greatest l.s.c. function $\leq R_u^A$. This is the <u>sweep</u> (or balayēe function) of u relatively to A:

$$\hat{R}_u^A(x) := \liminf_{y \to x} R_u^A(y) \qquad (x \in X) .$$

We have $\hat{R}_u^A \in {}^*H^+$ and obviously

$$0 \leq \hat{R}_u^A \leq R_u^A \leq u .$$

The initial interest leads then to the study of the base of A

$$b(A) := \bigcap_{u \in {}^*H^+} \{x \in X : \hat{R}_u^A(x) = u(x)\} .$$

It has the following fundamental properties:

$$\mathring{A} \subset b(A) \subset \overline{A} ,$$

$$b(A) = \{x \in X : \hat{R}_{u_0}^A(x) = u_0(x)\} \quad \text{for some } u_0 \in {}^*H^+ \cap C ,$$

in particular, $b(A)$ is a G_δ-set.

For every Radon measure $\mu \geq 0$ on X with compact support there exists a unique Radon measure $\mu^A \geq 0$ on X satisfying

$$\int u \, d\mu^A = \int \hat{R}_u^A \, d\mu \qquad \text{for all} \quad u \in {}^*H^+ .$$

μ^A is called the <u>swept out</u> of μ on A. It is carried by $\overline{A}$. By choosing for μ unit masses ε_x at points $x \in X$ it follows that

$$b(A) = \{x \in X : \varepsilon_x^A = \varepsilon_x\} .$$

We are now able to express the solution of the generalized Dirichlet problem in terms of balayage:

For every relatively compact open set U and every $f \in C(U^*)$ the solution $H^U f$ satisfies

$$H^U f(x) = \int f \, d\varepsilon_x^{\complement U} = \int f \, d\varepsilon_x^{U^*} \quad (x \in U) .$$

The set U_r of regular boundary points is given by

$$U_r = b(\complement U) \cap \overline{U} .$$

4. The weak Dirichlet problem

Again let U be a relatively compact open subset of X. The fact that a function $f \in C(U^*)$ may not admit an extension to a function $h \in H(U)$ led to the introduction of the generalized solution $H^U f$ which is a harmonic extension of f but is not necessarily continuous at all points of the boundary U^*.

Another way of turning the problem is the following: Are there at least some subsets B of the boundary such that every continuous function f on B admits a continuous extension to a function in $H(U)$? Because of a general minimum principle a natural candidate for such a set B would be the Choquet boundary $Ch_{H(U)}\overline{U}$ of $\overline{U}$ with respect to $H(U)$.

The <u>Choquet boundary</u> $Ch_{H(U)}\overline{U}$ is the set

$$Ch_{H(U)}\overline{U} := \{x \in \overline{U} : M_x(U) = \{\varepsilon_x\}\}$$

where

$$M_x(U) := \{\mu : \mu(h) = h(x) \quad \text{for all} \quad h \in H(U)\}$$

denotes the set of all <u>representing measures</u> for x (with respect to $H(U)$).

If for example V is regular, $\overline{V} \subset U$ and $x \in V$ then μ_x^V is a representing measure for x. More generally, for every $x \in \overline{U}$ the swept-out $\varepsilon_x^{\complement U}$ of ε_x on $\complement U$ is a representing measure for x. In particular, the Choquet boundary $Ch_{H(U)}\overline{U}$ is a subset of the set U_r of regular points. For the Laplace equation these two sets coincide whereas for the heat equation the Choquet boundary may be a proper subset of U_r.

We have the following minimum principle: For every $h \in H(U)$ there exists a point $z \in Ch_{H(U)}\overline{U}$ such that $h \geq h(z)$. In particular, if h_1, $h_2 \in H(U)$ and $h_1 = h_2$ on $Ch_{H(U)}\overline{U}$ then $h_1 = h_2$.

Thus the following weak Dirichlet problem arises: Given a compact subset K of $Ch_{H(U)}\overline{U}$ and a continuous function f on K, is there a continuous extension to a function in $H(U)$?

The solution of this problem is obtained by the following result.

Theorem ([2]). For every $x \in \overline{U}$ there exists a unique measure $\mu_x \in M_x(U)$ which is carried by $Ch_{H(U)}\overline{U}$. For every $x \in \overline{U} \setminus Ch_{H(U)}\overline{U}$,

$$\mu_x = \varepsilon_x^{Ch_{H(U)}\overline{U}} .$$

A very general reasoning now yields the following consequence.

Corollary. 1. The weak Dirichlet problem is solvable.
2. $\{\rho \in H(U)^* : \rho \geq 0, \rho(1) = 1\}$ is a simplex.

Furthermore, a close study of the Choquet boundary yields a characterization of $Ch_{H(U)}\overline{U}$ which is similar to the one obtained for U_r :

$$Ch_{H(U)}\overline{U} = \beta(\complement U) \cap \overline{U}$$

where $\beta(\complement U)$ is the greatest subset C of $\complement U$ such that $b(C) = C$.

5. General PWB-method

We shall now see that for every $x \in U$ the measure $\varepsilon_x^{Ch_{H(U)}\overline{U}}$ and many other representing measures can be obtained by a procedure in the spirit of Perron-Wiener-Brelot.

For every compact subset K of U^* let ${}^*H_K(U)$ be the set of all functions v which are limits of an increasing sequence (v_n) of l.s.c. real functions v_n on $\overline{U}$, hyperharmonic on U and continuous on $\overline{U} \setminus K$. Then ${}^*H_K(U)$ is a convex cone such that $H(U) \subset {}^*H_K(U) \subset {}^*H_{U^*}(U) = {}^*H(U)$.

Furthermore

$$Ch_{H(U)}\overline{U} \subset Ch_{{}^*H_K(U)}\overline{U} \subset K \cup Ch_{H(U)}\overline{U}$$

where the last inclusion is a consequence of the local characterization of the Choquet boundary. Indeed, obviously $Ch_{{}^*H_K(U)}\overline{U} \subset U^*$. So let $x \in U^* \setminus (K \cup Ch_{H(U)}\overline{U})$. Then there exists an open neighborhood V of x such that $\overline{V} \cap K = \emptyset$. Defining $W = U \cap V$ we have $x \in V \cap \complement \beta(\complement U) \subset \complement \beta(\complement W)$ and hence $x \notin Ch_{H(W)}\overline{W}$. Thus $\varepsilon_x^{Ch_{H(W)}\overline{W}} \neq \varepsilon_x$ and being a representing measure of x with respect to ${}^*H_\emptyset(W)$ the measure $\varepsilon_x^{Ch_{H(W)}\overline{W}}$ is a representing measure of x with respect to ${}^*H_K(U)$.

Let B be a Borel subset of U^* containing $Ch_{H(U)}\overline{U}$. Defining

$${}^*H_B(U) = \bigcup_{K \text{ cp.} \subset B} {}^*H_K(U)$$

we thus have the following minimum principle: If $v \in {}^*H_B(U)$ and $v \geq 0$ on B then $v \geq 0$ on $\overline{U}$.

Let $f \in C(U^*)$. Defining

$$\overline{H}_B^U f = \inf \{v \in {}^*H_B(U) : v \geq f \text{ on } B\} ,$$

$$\underline{H}_B^U f = \sup \{w \in - {}^*H_B(U) : w \leq f \text{ on } B\}$$

the minimum principle yields $\underline{H}_B^U f \leq \overline{H}_B^U f$. If there exists a function $h \in H(U)$ such that $h = f$ on U^* then evidently $h \leq \underline{H}_B^U f$ and $\overline{H}_B^U f \leq h$, hence $\overline{H}_B^U f = \underline{H}_B^U f = h$. In the general situation we have the following result.

Proposition ([3]). For every $f \in C(U^*)$

$$\overline{H}_B^U f = \underline{H}_B^U f =: H_B^U f ,$$

and $H_B^U f$ is harmonic on U. Furthermore, for every $x \in \overline{U} \setminus B$

$$H_B^U f(x) = \int f \, d\varepsilon_x^B .$$

Proof. It suffices to consider the case $f = v|_X$ for some continuous real $v \in {}^*H^+$. Then ${}^*H^+ \subset {}^*H_B(U)$ implies that

$$\overline{H}_B^U f|_{\overline{U}} \leq R_v^B .$$

Let K be a compact subset of B, $w = R_v^K|_{\overline{U}}$. Then $w \in - {}^*H_K(U), w \leq v$. Hence $R_v^K \leq \underline{H}_B^U f$ on $\overline{U}$. Taking the supremum of all R_v^K we obtain

$$R_v^B|_{\overline{U}} \leq \underline{H}_B^U f .$$

References

[1] Bauer, H.: Harmonische Räume und ihre Potentialtheorie. Lecture Notes in Mathematics 22. Berlin-Heidelberg-New York: Springer 1966

[2] Bliedtner, J., Hansen, W.: Simplicial Cones in Potential Theory. Inventiones math. 29, 83-110 (1975)

[3] Bliedtner, J., Hansen, W.: Simplicial Cones in Potential Theory II (Approximation Theorems). Inventiones math. 46, 255-276 (1978)

[4] Constantinescu, C., Cornea, A.: Potential Theory on Harmonic Spaces. Berlin-Heidelberg-New York: Springer 1972

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