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CONTROL AND THE VAN DER POL EQUATION

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1.

After playing a central rôle in the theory of nonlinear ordinary differential equations for over 50 years, Van der Pol's equation

$$(E_o) \qquad \ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

recently attracted the attention also from people working in control. This is accounted for by the fact that a system (electronic oscillator, living organism, or whatever else) governed by (E_o) cannot be brought to rest or to periodicity in a finite time by any change in the initial state $x(0)$, $\dot{x}(0)$. For that purpose one has to replace (E_o) by

$$(E_u) \qquad \ddot{x} + \mu(x^2 - 1)\dot{x} + x = u(t) \, ,$$

where $u : t \mapsto u(t)$ denotes some appropriate external force acting as a control.

In 1969 E. Ya.Roitenberg [6], as an application of a general theorem, proved that any solution of (E_u) can be brought to rest in a prescribed time T, provided that arbitrarily large controls u are allowed.

A more realistic approach, suggested already in Lee - Markus' book ([5], p. 391) of 1967, was adopted by Eleanor M. James in her 1972 PhD Thesis (published in a condensed version in 1974, [4]), where u is assumed to be bounded by some given $k > 0$ and T is not fixed in advance, which gives rise to the minimum time problem.

The same point of view was adopted in the 1976-77 Thesis of my pupil Gabriele Villari, [7], and, still more recently (1977), by N. K. Alekseev, [1].

It should also be noted that controllability and minimum time problems for an equation

$$\ddot{x} + g(x,\dot{x}) = u(t)$$

are studied in the books of Lee - Markus, [5], and Boltyanskii, [2], under assumptions on g which are not satisfied by $g(x,\dot{x}) = \mu(x^2-1)\dot{x}$.

To be more specific about (E_u) let us denote, as usual, by $L^\infty_{loc}(\mathbb{R})$ the class of measurable, locally essentially bounded functions $u : t \mapsto u(t)$, $t \in \mathbb{R}$, $u(t) \in \mathbb{R}$, and by U_k the set of "admissible" controls

$$U_k = \{ u \in L^\infty_{loc}(\mathbb{R}) : |u(t)| \leq k , \quad \text{a.e.} \quad t \in \mathbb{R} \}$$

for a given $k > 0$.

Also, denote by U'_k the subset of "relay" controls, i.e., the set of $u \in U_k$ taking only the two values $-k$, k, with a finite number of switches from one value to the other on every bounded interval.

If we write (E_u) in the equivalent form

$$(S_u) \qquad \begin{cases} \dot{x} = y \\ \dot{y} = -x + \mu y - \mu x^2 y + u(t), \end{cases}$$

we know that, for $u = 0$, (S_0) has a limit cycle Γ_μ.

We shall then consider :

Problem P_1. Find some $u \in U_k$ such that the corresponding solution of (S_u) joins Γ_μ with the rest point $0 = (0,0)$ in minimum time ;

Problem P_2. Find some $u \in U_k$ such that the corresponding solution of (S_u) goes from 0 to Γ_μ in minimum time.

A change of μ into $-\mu$ transforms one problem into the other. However we shall keep them distinct since we shall constantly assume $\mu > 0$.

2.

We shall deal first with problem P_1.

With fixed $\mu > 0$, $k > 0$, let us denote by $V(\mu,k)$ the set of points in the (x,y)-plane which can be transferred to 0 along the solutions of (S_u) by using $u \in U_k$. Let $V'(\mu,k)$ be the subset of $V(\mu,k)$ corresponding to U'_k.

According to [1], [4], [7], it can be shown that

(2.1) $$V'(\mu,k) = V(\mu,k)$$

is an open connected (not necessarily convex) set, symmetric with respect to 0 , containing the circle $x^2 + y^2 \leq k^2/\mu^2$.
As a consequence of the presence of the nonlinear term $\mu x^2 y$ in (S_u) there are pairs (μ,k) for which $V'(\mu,k) = \mathbb{R}^2$, so that we can consider the two sets

$$\mathcal{E} = \{(\mu,k) : V'(\mu,k) = \mathbb{R}^2\},$$
$$\mathcal{N} = \{(\mu,k) : V'(\mu,k) \neq \mathbb{R}^2\}.$$

When $(\mu,k) \in \mathcal{N}$, $V'(\mu,k)$ is bounded by an arc of an orbit of (S_k) (i.e., (S_u) with $u = k$) lying in the half-plane $y < 0$ and by the symmetric arc of an orbit of (S_{-k}) in the half-plane $y > 0$. A comparison of the vector fields defined by (S_k), (S_o) and (S_{-k}) shows that

$$(\mu,k) \in \mathcal{N} \iff V'(\mu,k) \text{ interior to } \Gamma_\mu$$

so that problem P_1 can have solutions only if $(\mu,k) \in \mathcal{E}$ and nothing changes if we replace U'_k by the larger set U_k , because of (2.1). Therefore it is important to recognize whether a given (μ,k) belongs to $\mathcal{E}$ or to $\mathcal{N}$.

Now, for every $\mu > 0$ there exists

(2.2) $$k^*(\mu) = \max\{k : (\mu,k) \in \mathcal{N}\}$$

so that

$$\mathcal{E} = \{(\mu,k) : 0 < \mu,\ k^*(\mu) < k\},$$
$$\mathcal{N} = \{(\mu,k) : 0 < \mu,\ k \leq k^*(\mu)\},$$

but, unfortunately, no explicit formula giving the value of $k^*(\mu)$ for each $\mu > 0$ is known.

All that is known (again according to [1] , [4] , [7] , with some improvements) about k^* can be summarized as follows :

$$k^*(\mu) \leq \min\{\sqrt{3}\,\mu,\ 1\} , \quad 0 < \mu ,$$

$$\sqrt{1 - 2/\mu} < k^*(\mu) , \quad 2 \leq \mu ,$$

$$4\,\frac{\mu}{\sqrt{\mu+1}}\;\frac{\mu_2}{(3\mu_1)^{3/2}} \leq k^*(\mu)\ ,\qquad 0<\mu\ ,$$

where

$$\mu_1 = (\mu^2+4+\mu\sqrt{\mu^2+4})/2\quad,\quad \mu_2 = (\mu^2+4-\mu\sqrt{\mu^2+4})/2\ .$$

Consequently,

$$\lim_{\mu\to 0} k^*(\mu) = 0\ ,\qquad \lim_{\mu\to+\infty} k^*(\mu) = 1\ ,$$

$$\frac{2\sqrt{2}}{9}\sqrt{3} \leq \liminf_{\mu\to 0}\frac{k^*(\mu)}{\mu} \leq \limsup_{\mu\to 0}\frac{k^*(\mu)}{\mu} \leq \sqrt{3}\ .$$

If we define $k^*(0) = 0$ then k^* turns out to be lipschitzian on $[0,\bar{\mu}]$ for every $\bar{\mu} > 0$ ([1]).

In the absence of an explicit representation of $k^*(\mu)$ it might be interesting to ascertain whether k^* is an increasing function and how much regular it is : for instance, $k^* \in C^{(1)}(\mathbb{R}_+)$?

More information about k^* probably could be obtained by studying the behavior of the limit cycles of the systems (S_k), (S_{-k}). It can be shown that the singular point $K = (k,0)$ is a global attractor for (S_k) if $k \geq 1$, whereas for $0 \leq k < 1$ there is at least one limit cycle $\Gamma_{\mu,k}$ of (S_k) around K . To decide whether $\Gamma_{\mu,k}$ is unique some ad hoc proof has to be found since the usual techniques fail for $k > 0$ because of the lack of symmetry of the orbits of (S_k) with respect to K . Taking the uniqueness for granted, we have

$$k \leq k^*(\mu) \iff V'(\mu,k) \subset G(\mu,k)\ ,$$

where $G(\mu,k)$ is the intersection of the two regions interior to $\Gamma_{\mu,k}$ and its symmetric $\Gamma_{\mu,-k}$ with respect to 0 .

3.

Introducing $r^2 = x^2 + y^2$ we have

$$r\,\dot{r} = \mu(1-x^2)\,y^2 + y\,u(t) \leq \mu r^2 + k\,r$$

along the solutions of (S_u) with $u \in U_k$, hence

$$r(t) \leq (k/\mu + r(0))\,e^{\mu t} - k/\mu\ ,$$

i.e., there exists a uniform bound for all solutions initiating at (x^o,y^o) for a finite time duration. Consequently (Cf. E. B. Lee - L. Markus, [5] , Th.4 , p. 259) we can go from any point $(x^o,y^o) \in V(\mu,k)$ to 0 by means of $u \in U_k$ in a minimum time $T(x^o,y^o)$.

Further, the function $T : (x^o,y^o) \mapsto T(x^o,y^o)$ is lower semicontinuous on $V(\mu,k)$ so that, if $(\mu,k) \in \mathcal{E}$,it takes its minimum value $T_{\mu,k}$ on the compact set Γ_μ .

Therefore, problem P_1 has solutions for every $(\mu,k) \in \mathcal{E}$.

To determine such solutions one can use the techniques derived from Pontryagin's maximum principle (Cf. Lee- Markus' book, Chapter 7). In fact, if $(x,y) : t \mapsto (x(t),y(t))$ is the solution of (S_u) , $x(0)=x^o$, $y(0) = y^o$, corresponding to a minimizing control u ,then there exists a solution $(\eta_1,\eta_2) : t \mapsto (\eta_1(t),\eta_2(t))$ of the linear system

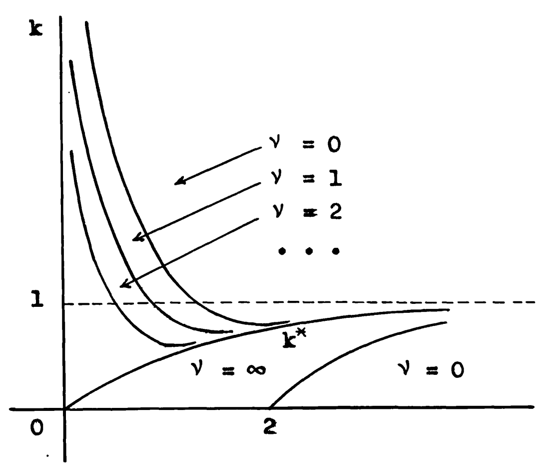
$$\begin{cases} \dot{\eta}_1 = (1 + 2\mu\, x(t)\, y(t))\, \eta_2 \\ \dot{\eta}_2 = -\eta_1 + \mu\,(x^2(t) - 1)\eta_2 \end{cases}$$

such that $\eta_2(t)\, u(t) = \max\limits_{|v| \leq k} \eta_2(t)\, v$, so that

$$u(t) = k \operatorname{sign} \eta_2(t) \, .$$

Therefore, minimum time controls are of relay type.

The maximum number ν of switches depends on (μ,k) according to the map shown in Fig. 1.



Such map is obtained by the construction of the switching locus by a combination of geometrical, comparison and computational methods (See [4] , [7]).

Unfortunately, no explicit formulas are known to represent the "hyperbolas" in the $\mathcal{E}$ region.

4.

The next question is that of locating the points of Γ_μ at which the minimum $T_{\mu,k}$ is attained. This is a difficult question, because no analytical representation of Γ_μ is presently (1977) known, so we cannot expect to have exact solutions. On the other hand Γ_μ can be enclosed within an annulus whose inner and outer boundaries have simple enough analytical representations and may be made satisfactorily close to Γ_μ (Cf. R. Gomory - D. E. Richmond, [3]). This, and the fact that also a good approximation of the switching locus can be obtained, suggest that substantial aid to the location of minimizing points can be expected from numerical methods.

The transversality condition is also of some help. In our case, such condition means that the vector of components $\eta_1(0)$, $\eta_2(0)$ is orthogonal to the tangent vector to Γ_μ at a minimizing point $M = (x,y)$, so that

$$\eta_1(0)\, y + \eta_2(0)\left[-x + \mu y - \mu x^2 y\right] = 0 \ . \tag{4.1}$$

Therefore the points M are among the intersections of Γ_μ with the cubic (4.1). Since Γ_μ can be locally represented by an analytic function $x \mapsto y(x)$ or $y \mapsto x(y)$, like every other orbit of (S_0), the number of intersections is finite. It is an open question whether there can be more than one pair of (symmetric) intersections.

5.

To deal with problem P_2 one has to replace $V(\mu,k)$ by the set $W(\mu,k)$ of (x,y) points which can be attained from 0 along the solutions of (S_u) by using $u \in U_k$. Correspondingly, $V'(\mu,k)$ is replaced by $W'(\mu,k)$ and it can be shown that

$$W'(\mu,k) = W(\mu,k)$$

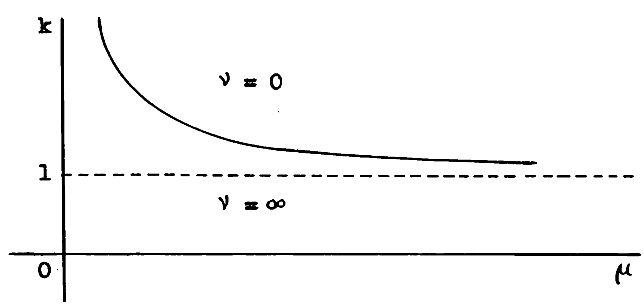
is an open connected set symmetric with respect to 0 . The effect of the term $\mu x^2 y$ in (S_u) is that $W(\mu,k)$, unlike $V(\mu,k)$, is bounded for <u>all</u> pairs $\mu > 0$, $k > 0$, whereas, in the absence of such term, the corresponding set $W(\mu,k)$ would be $= \mathbb{R}^2$. However,

$$\Gamma_\mu \subset W'(\mu,k) \ , \mu > 0, \ k > 0 \ ,$$

and by the Weierstrass - Baire theorem we see that <u>problem P_2 has solutions for all pairs</u> $\mu > 0$, $k > 0$.

To determine the solutions offers the same difficulties as in the case of problem P_1.

The construction of the switching locus shows that, depending on μ, k , either one can go from 0 to any point in $W'(\mu,k)$ in minimum time with one switch at most, or, for every positive integer N there are points in $W'(\mu,k)$ such that the corresponding number of switches is $>$ N.



The map in Fig. 2 shows the dependence of the maximum number ν of switches on μ, k .

Again, no explicit representation of the "hyperbola" separating the two zones is known.

The number of minimizing points, in pairs, is still finite, but uniqueness and their location on Γ_μ are open questions.

R e f e r e n c e s.

[1] N. K. Alekseev, Some questions of controllability for two-dimensional systems (Russian), Diff. Uravnenyia, 13 (1977), 387-397 ;

[2] V. G. Boltyanskii, Mathematical methods of optimal control, Holt-Rinehart-Winston, 1971 ;

[3] R. Gomory - D. E. Richmond, Boundaries for the limit cycle of Van der Pol's equation, Quart. Appl. Math., 9 (1951), 205 - 209 ;

[4] E. M. James, Time optimal control and the Van der Pol oscillator, J. Inst. Maths. Appls., 13 (1974), 67-81 ;

[5] E. B. Lee - L. Markus, Foundations of optimal control theory, J. Wiley and Sons, 1967 ;

[6] E. Ya. Roitenberg, On a sufficient condition of controllability for nonlinear systems (Russian), Vestnik Moskov. Univ., 1 (1969), 28-33 ;

[7] Gabriele Villari, Controllo del ciclo di Van der Pol, Tesi di Laurea in Matematica, Univ. di Firenze, 1976-77.

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