EQUADIFF 4

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Asymptotic invariant sets of autonomous differential equations

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ASYMPTOTIC INVARIANT SETS OF AUTONOMOUS DIFFERENTIAL EQUATIONS

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Let us suppose that the solutions of the real autonomous system

(1) $\dot{x} = f(x), \quad x = (x_1,\ldots,x_n), \quad \dot{} = \frac{d}{dt}$

are, in a domain D of R_n, uniquely determined by their initial values and exist for all t. Then the whole D is an invariant set of (1), but this is of no interest. We look for nontrivial invariant sets forming some interesting surfaces - perhaps certain curves - or investigate how the invariant surfaces of the linear equation

(2) $\dot{x} = Ax, \quad A = (a_{ik})$

will be deformed into the corresponding invariant surfaces of the nonlinear (perturbed) equation

(3) $\dot{x} = Ax + f(x), \quad F = (f_1,\ldots,f_n), \quad f_i = f_i(x)$.

So we can seek asymptotically invariant surfaces, too, i.e. such invariant surfaces of (2) to which the corresponding invariant surface of (3) tends as $t \to \infty$. In a paper written jointly with A. Elbert [1] - restricted to n=3 and A = const - a number of such problems were solved. We were faced there with the problem: The full set of paths of (3) depends on two parameters which need not be specified in detail - say u and v - both of which depend on three parameters X_0 , Y_0 , Z_0

$$u = u(X_0, Y_0, Z_0), \quad v = v(X_0, Y_0, Z_0)$$

where

$$X_0 = \lim_{t\to\infty} xe^{-\lambda t}, \quad Y_0 = \lim_{t\to\infty} (ye^{-\lambda t} - X_0 t),$$

$$Z_0 = \lim_{t\to\infty} (ze^{-\lambda t} - Y_0 t - \tfrac{1}{2} X_0 t^2) .$$

These are the "end values" of the solutions which - conversely - determine them uniquely by means of the corresponding integral equations provided some appropriate supplementary conditions are introduced. - Now putting $X_0 = 0$ it is plausible, however it must be proved, that it arises a one parameter family of paths, i.e. a surface. In the work referred to above this was done and the unique parameter (Z_0/Y_0') upon which the family depended was determined as well as the corresponding invariant surface. Here Y_0' means the value of Y_0 obtained by putting $X_0 = 0$.

In this lecture we give an example of an asymptotically invariant surface. Assume now in (2) - (3) n=3,

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix} \qquad \lambda < 0 ,$$

$$F = (f,g,h), \quad |f|,|g|,|h| \le r\omega(r), \quad r = \sqrt{x^2 + y^2 + z^2} ,$$

$$r < r_1 , \quad r_1 > 0$$

where $\omega(r)$ is nondecreasing continuous, $\omega(0) = 0$ and

$$\int_{+0} \frac{\omega(r)}{r} (\log r)^4 dr < \infty .$$

Let us determine all the quadratic invariant surfaces $\rho = 0$ of (2), where $\rho = x^*Bx$ is a quadratic form with $B = (b_{ik})$, b_{ik} = const, $b_{ik} = b_{ki}$. The solutions of (2) are

$$x = x_0 e^{\lambda t}, \quad y = (y_0 + x_0 t) e^{\lambda t}, \quad z = (z_0 + y_0 t + \tfrac{1}{2} x_0 t^2) e^{\lambda t}$$

which have to satisfy $\rho = 0$ for every t. This condition gives necessarily

$$\rho \equiv ax^2 + b(y^2 - 2xz) \qquad (a^2 + b^2 > 0)$$

where a and b are arbitrary parameters. Thus the invariant surfaces of (2) in question are

$$S(a,b) : \quad \rho = 0$$

and an easy consideration shows that these are conical surfaces (see Fig. 1) with the origin as vertex, symmetric with respect to

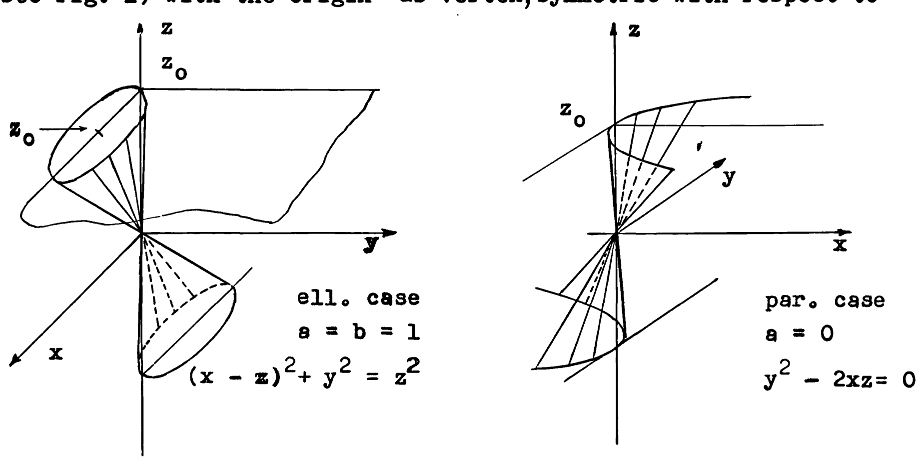

Fig. 1a　　　　Fig. 1b

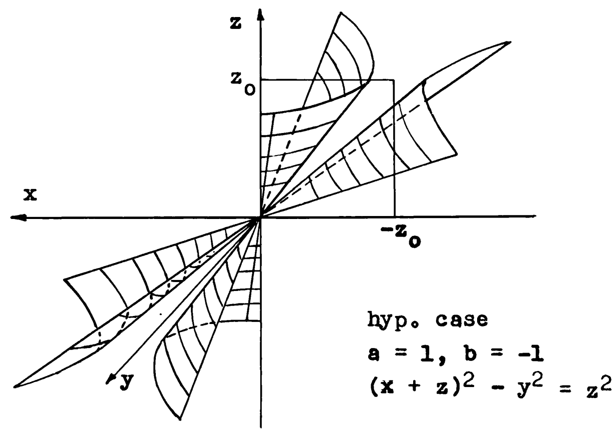

Fig. 1c

the plane (yz) and elliptic, hyperbolic or parabolic according to ab $\gtrless 0$ or a=0 . Every element of S(a,b) contains the axis z which is an invariant line itself. Similarly, the plane (yz) is an invariant plane. Every point $P_o(X_o ,Y_o ,Z_o)$ of D - except the points of the axis z - is crossed by a member of the family S(a,b) the parameter a/b or b/a of which can be uniquely determined from

$$(3') \qquad \rho_o = ax_o^2 + b(y_o^2 - 2x_oz_o) = 0$$

or otherwise expressed. a and b can be uniquely determined from (3') up to a common factor. Also we can state that every path lies in a single member of S(a,b). By means of the integral equations - which we do not write here explicitly - it can be easily proved that the triple (X_o ,Y_o ,Z_o) and the triple

$$(4) \qquad \begin{aligned} X &= X_oe^{\lambda t_1}, \\ Y &= (Y_o + X_ot_1)e^{\lambda t_1}, \qquad (t_1 \geq 0) \\ Z &= (Z_o + Y_ot_1 + \tfrac{1}{2} X_ot_1^2)e^{\lambda t_1} \end{aligned}$$

(taken as end values) determine the same path of (3). They have only a shift of parameter t_1 with respect to each other. However, in the space (X,Y,Z) (4) is the parametric equation of a curve σ . Thus the path p of (3) and the curves σ are one to one. Since

$$aX^2 + b(Y^2-2XZ) = \left[aX_0^2 + b(Y_0^2-2X_0Z_0)\right]e^{2\lambda t_1},$$

the surface

(5) $$\textstyle\sum: aX^2 + b(Y^2 - 2XZ) = 0$$

in this space is formed by the curves σ provided b/a is determined from

(6) $$aX_0^2 + b(Y_0^2 - 2X_0Z_0) = 0 .$$

Then the corresponding paths p of (3) form an invariant surface S' of (3). By (6) and the asymptotic form (not given here) of the solutions of (3) we have

$$ax^2 + b(y^2-2xz) = \left[aX_0^2 + b(Y_0^2-2X_0Z_0) + o(1)\right]e^{2\lambda t} = o(e^{2\lambda t}), \quad t\to\infty$$

$$ax^2 + b(y^2-2xz) = \left[aX_0^2 + b(Y_0^2-2X_0Z_0) + o(1)\right]e^{2\lambda t} = o\left[\frac{r^2}{(\log r)^4}\right], \quad r\to 0 .$$

The last expression is a consequence of the asymptotic formula $r \sim t^2e^{\lambda t}$. The asymptotic invariant surface of (3) belonging to p which has the end values X_0, Y_0, Z_0 is

(7) $$ax^2 + b(y^2 - 2xz) = 0$$

where a and b are given by (6) and S' is situated between the surfaces

(8) $$ax^2 + b(y^2-2xz) = \pm F(x,y,z), \quad F = o\left[\frac{r^2}{(\log r)^4}\right], \quad r\to 0$$

(where F is not determined in more detail) and approaches (7) as $t\to\infty$ which is an invariant surface of (2).

Reference

[1] I.Bihari, A.Elbert: Perturbation theory of three-dimensional real autonomous systems, Periodica Math. Hung. 4 (4), (1973), pp. 233-302

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