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SOLUTIONS OF A LINEAR SECOND ORDER EQUATION OF PARABOLIC TYPE DEFINED IN AN UNBOUNDED DOMAIN

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1

The following linear parabolic equation is considered:

$$(1)\qquad F(u) \equiv \sum_{i,j=1}^{m} a_{ij}(t, X)\, u''_{x_i x_j} + \sum_{k=1}^{m} b_k(t, X)\, u'_{x_k} + c(t, X)\, u - u'_t = f(t, X)$$

$$(X = (x_1, \ldots, x_m);\; a_{ij} = a_{ji},\; i, j = 1, 2, \ldots, m),$$

whose coefficients and the right hand side member $f(t, X)$ are defined in the zone Σ: $0 < t < T$, $X \in E^m$; E^m being the m-dimensional Euclidean space. T may equal infinity. In what follows, we shall denote the set $\langle 0, T) \times E^m$ by $\widetilde{\Sigma}$. The quadratic form

$$\mathfrak{A}(\Lambda) = \sum_{i,j=1}^{m} a_{ij}(t, X)\, \lambda_i \lambda_j\,, \quad (t, X) \in \Sigma\,,$$

is assumed to be positive definite.

We shall say that a function $u(t, X)$ is regular in a set G of the time-space E^{m+1} of the variables $t, x_1, \ldots, x_m$, if $u(t, X)$ is continuous in the set G and has continuous derivatives $u'_{x_i}, u''_{x_i x_j}, u'_t$ $(i, j = 1, \ldots, m)$ in the interior $G^{(i)}$ of the set G; $u(t, X)$ is said to be a solution of equation (1) regular in G if the function is regular in the set G and satisfies equation (1) in the interior $G^{(i)}$.

We shall discuss the problem of finding a solution of equation (1) regular in the zone $\widetilde{\Sigma}$ and satisfying the initial condition

$$(2)\qquad u(0, X) = \varphi(X) \quad \text{for} \quad X \in E^m\,,$$

where $\varphi(X)$ is a given function continuous in E^m. This problem is often called the Cauchy problem but it is not the Cauchy problem in the sense which is attributed to this term in the general theory of equations with partial derivatives, because the initial conditions (Cauchy conditions) are reduced to a single one which is imposed on the characteristic, while for the case of a second order equation in the group of Cauchy conditions there are two conditions imposed on a manifold which is not a characteristic.

Most of the results we will discuss are transferred for the case where the coefficients and the right hand side $f(t, X)$ of equation (1) are defined on a certain domain D bounded by two m-dimensional domains S_0 and S_T lying on characteristics $t = 0$ and $t = T\,(T > 0)$ and by a certain lateral surface σ with the time orientation with regard to equation (1), the domain D being unbounded in the direction of the x_i-axis, i.e.

the function $|X|^2 = \sum_{i=1}^{m} x_i^2$ is unbounded on the set of points $(t, X) \in D$ (T may equal infinity; then we assume that the part of D situated in a certain zone $(0, T') \times E^m$ is unbounded in the direction of the x_i-axis). In this case we have to do with Fourier's problem in which boundary conditions are imposed on the surface σ in addition to the initial condition on S_0.

In the case when the values of the solution of equation (1) are given on surface σ (boundary condition of Dirichlet type) we have the first Fourier problem. Furthermore the boundary condition may merely be that at points of σ the values of the derivative of the solution in a certain direction l (in general depending on situation of the point) entering into the domain D are given. Then we have the second Fourier problem. At last, the following boundary condition may occur

(3)
$$\alpha(t, X)\frac{du}{dl} + \beta(t, X)\, u = g(t, X) \quad \text{for} \quad (t, X) \in \sigma,$$

where $\alpha(t, X)$, $\beta(t, X)$, $g(t, X)$ are functions defined on σ, $\alpha(t, X)$ being non-negative.

2

We say that a function $w(t, X)$ continuous in a set G of the time-space E^{m+1}, unbounded in the direction of the x_i-axis (cf. sec. 1), is of class E_α ($\alpha > 0$) if there exist two non-negative numbers M and K (generally dependent on the function $w(t, X)$ itself) such that the inequality

(4)
$$|w(t, X)| \leqq M \exp K|X|^\alpha$$

holds for $(t, X) \in G$.

A. N. Tihonov (see [22]) proved that, in a particular case of the heat equation

(5)
$$u''_{xx} - u'_t = 0,$$

the Cauchy problem, which has just been formulated, has at most one solution in class E_2. On the contrary, for every $\varepsilon > 0$ in class $E_{2+\varepsilon}$ there exists a solution of equation (5) regular in a zone $0 \leqq t \leqq T$, $-\infty < x < \infty$, vanishing for $t = 0$, and not vanishing identically in the zone.

If the function $\varphi(x)$ belongs to the class E_2, then the solution of the Cauchy problem for equation (5) with initial condition (2) (for $X = x$) is determined by the formula

(6)
$$u(t, x) = \frac{1}{2\sqrt{(\pi t)}} \int_{-\infty}^{\infty} \varphi(y) \exp\left[-\frac{(x-y)^2}{4t}\right] dy$$

(see [7], p. 300). The integral on the right hand side is the Fourier-Poisson's integral. Formula (6) determines the solution of the Cauchy problem for a certain zone whose height depends on the number K appearing in the inequality (4) ($\alpha = 2$, the definition of class E_2). This height may be infinite. In particular, when the function $\varphi(x)$ is of class E_1, the height is infinite, that is, the solution is defined in the half-space $t \geqq 0$.

Notice that earlier, E. Holmgren (see [9]) proved the uniqueness of the solution of the Cauchy problem for equation (5) in a class of functions wider than E_2 as concerns the growth condition for $x \to \infty$, but under stronger assumptions concerning the regularity of these functions. S. Täklind (see [21]) showed that in order to hold the uniqueness of the solution of Cauchy problem in the class of functions satisfying the inequality

$$|u(t, x)| \leqq M \exp [K|x| \, h(|x|)] ,$$

where $h(z)$ is a positive function for $z > 0$, it is necessary and sufficient that the integral

$$\int_1^\infty \frac{dz}{\bar{h}(z)}$$

diverges, $\bar{h}(z)$ being the greatest non-decreasing minorant of $h(z)$.

Tihonov's result concerning the uniqueness of Cauchy problem in class E_2 has been extended by the author of the present report (see [10] and [11]) to the general linear normal parabolic equation of the form (1), under the assumption that the coefficients of equation (1) are bounded. This result can also be immediately applied to the first Fourier problem in the domain unbounded in the direction of the x_i-axis (see sec. 1). To prove the uniqueness of the solution of this problem we make use of a general theorem of M. Picone (see [16]). This theorem is the following

Theorem 1. (*M. Picone.*) *It is assumed that there exists a functions $H(t, X)$ regular and positive in the set $\tilde{D} = D + S_0 + \sigma$ (in the set $\tilde{\Sigma}$ for the case of Cauchy problem), and satisfying the inequality $F(H) \leqq 0$ in the domain D (in the zone Σ). Then the trivial solution $u(t, X) \equiv 0$ is a unique solution of equation* (1) (*with $f(t, X) \equiv 0$*) *regular in $\tilde{D}$ (in $\tilde{\Sigma}$), vanishing on the set $\tilde{S} = S_0 + \sigma$ (vanishing for $t = 0$), and satisfying the condition*

(7) $$\lim_{|X| \to \infty} \frac{u(t, X)}{H(t, X)} = 0 .$$

Because of the role of the function $H(t, X)$ in the theorem, we shall call it the *stifling divisor.*

To prove the uniqueness of the solution of the first Fourier problem and, in particular, of the Cauchy problem in class E_2 we choose the stifling divisor of the following form

(8) $$H(t, X; k) = \exp \left[\frac{k}{1 - \mu(k) \, t} \sum_{i=1}^{m} x_i^2 + \nu(k) \, t \right]$$

in which a certain parameter k and functions $\mu(k)$ and $\nu(k)$ of the parameter appear. One chooses the value of the parameter $k > K$, K being the constant appearing in the definition of class E_2 (see (4)).

The above result applies to the case of coefficients satisfying the conditions

$$(9) \qquad |a_{ij}(t, X)| \leqq A_0, \ |b_j(t, X)| \leqq A_1|X| + B_1, \ c(t, X) \leqq A_2|X|^2 + B_2 .$$

See [12] for the details.

The theorem similar to theorem 1 concerning the second and third Fourier problems is also true. Applying this theorem the author of the report proved the uniqueness of the solution of the second and third Fourier problems in class E_2 for some particular domains unbounded in the direction of the x_i-axis under the assumption of boundedness for the coefficients of equation (1) and condition (3). See [13] for details. As concerns generalization to the case of the coefficients of equation (1) fulfilling condition (9), certain results have been obtained by P. Besala (see [2]) and by I. Łojczyk-Królikiewicz (to appear in Annales Polonici Mathematici).

3

The introduction of the stifling divisor also enables one to prove the existence of the solution of the Cauchy problem and Fourier problems for domains unbounded in the direction of the x_i-axis under the assumption that there exist solutions of convenient Fourier problems for a certain sequence of bounded domains. At present we shall occupy ourselves with the Cauchy problem and the first Fourier problem.

We assume the following hypothesis:

Hypothesis (A). The domain D (or the zone Σ) is assumed to be the sum of a monotonic increasing sequence of domains D_n ($D_n \subset D_{n+1}$) separated from domain D by surfaces $\mathscr{S}_n$ ($n = 1, 2, \ldots$) of which the distance from the origin tends to infinity together with n; the domains D_n being regular with respect to the first Fourier problem for equation (1).*)

One can prove the following

Theorem 2. *Suppose hypothesis* (A) *is true. Assume furthermore that* 1) *the coefficients of equation* (1) *are bounded in the domain* D *(in the zone* Σ*), or, at least, they satisfy the conditions* (9); 2) *the function* $f(t, X)$ *is of class* E_2 *in this set*; 3) $\Phi(t, X)$ *is a given function continuous and of class* E_2 *on the set* $\tilde{S}$ *(see theorem 1)* ($\varphi(x)$ *is a given function continuous and of class* E_2 *in the space* E^m); 4) *the height* T *of domain* $\tilde{D}$ *(of zone* $\tilde{\Sigma}$*) is equal to or less than a certain number dependent on the coefficients of* (1) *and on the functions* $f(t, X)$ *and* $\Phi(t, X)$ ($\varphi(X)$).

Under these assumptions there exists a solution $u(t, X)$ *of equation* (1) *regular in the domain* $\tilde{D}$ *(in the zone* $\tilde{\Sigma}$*) and satisfying the boundary condition*

$$u(t, X) = \Phi(t, X) \quad \text{for} \quad (t, X) \in \tilde{S}$$

*) A given domain is said to be regular with respect to the first Fourier problem for equation (1) if this problem always has a solution provided the functions appearing in the boundary and initial conditions are continuous.

(the initial condition (2)*). This solution is of class* E_2 *in every zone* $\langle 0, T' \rangle \times E^m$, *where* $0 < T' < T$, *and constitutes the limit of the sequence* $\{u_n(t, X)\}$ *of solutions of the first Fourier problem for equation* (1) *in the domains* D_n, *with the conditions*

$$u_n(t, X) = \bar{\Phi}(t, X) \quad \text{for} \quad (t, X) \in S_0 \,.\, \bar{D}_n + \sigma_n\,,$$

where σ_n *are the lateral surfaces of domains* D_n, *and* $\bar{\Phi}(t, X)$ *is a continuous extension of the function* $\Phi(t, X)$ *to the set D.*

In the proof of this theorem the stifling divisor (8) is used. For the details see [10] and [11].

A similar theorem concerning the second and third Fourier problems can also be proved, at least, for certain particular domains unbounded in the direction of the x_i-axis.

The above method for proving the existence of solutions of boundary-value problems in the unbounded domains can also be applied to boundary problems for equations of the elliptic type (see [14]). A similar method was used by L. Amerio (see [1]).

4

The uniqueness and existence theorems in class E_2, about which we have spoken above, have recently been extended by P. Besala (see [2]; the details are inserted in papers which will appear in Annales Polonici Mathematici vol. XIII) to the system of parabolic equations of the form

$$\frac{\partial u_h}{\partial t} = F_h\left(t, X, u_i, \frac{\partial u_h}{\partial x_j}, \frac{\partial^2 u_h}{\partial x_j\, \partial x_k}\right) \qquad (10)$$

$$(h, i = 1, \ldots, n; \quad j, k = 1, \ldots, m)\,.$$

It is assumed that every function F_h $(h = 1, \ldots, n)$ satisfies the following condition: for $y_h \geqq \bar{y}_h$ we have

$$F_h(t, X, y_i, z_j, z_{jk}) - F_h(t, X, \bar{y}_i, \bar{z}_j, \bar{z}_{jk}) \leqq \qquad (11)$$

$$\leqq L_0 \sum_{j,k=1}^{m} |z_{jk} - \bar{z}_{jk}| + (L_1|X| + L_2) \sum_{j=1}^{m} |z_j - \bar{z}_j| + (L_3|X|^2 + L_4) \sum_{i=1}^{n} |y_i - \bar{y}_i|\,.$$

5

In the methods discussed the most essential role is attributed to the properties of solutions of parabolic equations called the extremum principle. These methods do not apply immediately to parabolic equations of higher orders and parabolic systems of a more general form. For these other methods are applied. For the details see e.g. the papers of S. Eĭdel'man [3], [4], [5], H. Grużewska [8], O. Ladyženskaya [15], W. Pogorzelski [17], L. Slobodeckiĭ [19], [20] and Ya. Žitomirskiĭ [24].

In Eĭdel'man's paper [4] the nonlinear system

$$\frac{\partial u}{\partial t} = F\left(t, X, u, \ldots, \frac{\partial^{2b} u}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}\right) \tag{12}$$

($u = (u_1, \ldots, u_n)$ being the vector function) parabolic in the sense of Petrovskiĭ has been considered and the uniqueness of the solution of the Cauchy problem has been proved in the class of functions which have bounded and Hölder continuous derivatives up to the $4b$-th order. The existence of the solution of this problem has been proved in the case when the functions appearing in the initial condition have derivatives which are Hölder continuous and bounded up to the $4b$-th order. On the right hand members one makes certain assumptions concerning the regularity which are not given precisely in this report.

If system (12) is almost linear i. e. the right hand side is linear with regard to the derivatives of order $2b$ and the coefficients of these derivatives depend only on the independent variables, then the uniqueness theorem holds true in class E_q, where $q = 2b/(2b-1)$, provided the coefficients of the derivatives of order $2b$ possess derivatives up to the $2b$-th order which are bounded and Hölder continuous with respect to the variables x_i and under other assumptions previously made concerning system (12). The existence theorem has also been proved under the assumption that the function $\varphi(X)$ is of class E_q and under certain assumptions on the coefficients which we do not give here in detail.

L. Slobodeckiĭ (see [20]) has considered a linear system of the form (12) (the right hand members are linear with regard to the unknown functions and their derivatives), parabolic in the sense of Petrovskiĭ for $0 \leqq t \leqq T$, and he has proved the uniqueness of the solution of the Cauchy problem in the class I_q of functions satisfying the condition: there exists a non-negative constant K such that the integral

$$\int_0^T \mathrm{d}t \int_{E^m} |u(t, X)| \exp\left[-K|X|^q\right] \mathrm{d}X$$

converges. The existence of the solution of this problem has also been proved under the assumption that the function $\varphi(X)$ satisfies the following condition: there exists a non-negative constant K such that the integral

$$\int_{E^m} |\varphi(x)| \exp\left[-K|X|^q\right] \mathrm{d}X$$

converges.*)

*) In Slobodeckiĭ's paper the initial condition (2) is formulated in another form.

6

A theorem of D. Widder [23] initiated a new direction of investigations concerning the uniqueness of the solution of the Cauchy problem for parabolic equations. According to the Widder's theorem the solution of heat equation (5) regular and non-negative in a zone $\tilde{\Sigma}$ (see sec. 1), vanishing for $t = 0$, vanish identically in this zone.

The result of Widder has been generalized on equation (1) successively by J. Serrin [18] (the case of two independent variables, the coefficients depend only on the variable x) and by A. Friedman [6]. We quote the theorem of Friedman.

Theorem 3. *The coefficients and the right side $f(t, X)$ of equation* (1) *are assumed to be defined in the zone $\tilde{\Sigma}$ and to satisfy the following conditions*

1) *there exists a positive number A_0 such that the inequality*

$$\mathfrak{A}(\Lambda) \equiv \sum_{i,j=1}^{m} a_{ij}(t, X)\, \lambda_i \lambda_j \geqq A_0 |\vec{\Lambda}|^2 = A_0 \sum_{j=1}^{m} \lambda_j^2$$

holds for every vector $\vec{\Lambda}(\lambda_1, \ldots, \lambda_m)$ and at every point of the zone;

2) *the functions a_{ij}, $\partial a_{ij}/\partial x_r$, $\partial^2 a_{ij}/\partial x_r \partial x_s$, $\partial a_{ij}/\partial t$, b_k, $\partial b_k/\partial x_r$, c $(i, j, k, r, s = 1, \ldots, m)$ are Hölder continuous and bounded in the zone $\tilde{\Sigma}$;*

3) *we have $f(t, X) \equiv 0$ in $\tilde{\Sigma}$.*

Under these assumptions let $u(t, X)$ be a solution of equation (1) *regular and non-negative in the zone $\tilde{\Sigma}$. If $u(0, X) = 0$ for $X \in E^m$, then $u(t, X) \equiv 0$ in $\tilde{\Sigma}$.*

It should be noted that the uniqueness of the solution of the Cauchy problem in the class of non-negative functions does not follow from Friedman's theorem but it can be deduced from the following theorems on which the proof of Friedman's theorem is also based.

Theorem 4. *Suppose the assumptions* 1)–3) *of theorem* 3 *hold true. If $u(t, X)$ is a solution of equation* (1) *regular in the zone $\tilde{\Sigma}$, belonging to the class I_2 (see sec. 5) and vanishing for $t = 0$, then $u(t, X) \equiv 0$ in $\tilde{\Sigma}$.**)

Theorem 5. *Under the assumptions* 1)–3) *of theorem 3, each solution of equation (1) regular and non-negative in the zone $\tilde{\Sigma}$ belongs to the class I_2.*

Finally I should like to present a recent result, of P. Besala and the author of the report, which has not yet been published.

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*) Evidently from theorem 4 the uniqueness of the solution of the Cauchy problem for equation (1) in class I_2 follows. This uniqueness has also been proved in the paper of Slobodeckiĭ. Friedman and Slobodeckiĭ evidently proved it independently of each other for the parabolic systems in the sense of Petrovskiĭ of the second and higher orders (in class I_q).

We shall say that a function $w(t,X)$ continuous in a set G of the time-space E^{m+1} is of class $\underline{E}_\alpha$ (of class $\bar{E}_\alpha$) if there exist two non-negative numbers M and K such that the inequality $w(t, X) \geqq - M \exp [K|X|^\alpha]$ ($w(t, X) \leqq M \exp [K|X|^\alpha]$ respectively) holds for the set G.

Now the following theorem holds true.

Theorem 6. *We suppose that hypothesis (A) relating to equation (1) holds and assumptions 1) and 2) of theorem 3 (of Friedman) concerning the coefficients of the equation are fulfilled. Furthermore we assume that the function $f(t, X)$ is continuous and of class E_2 in the zone $\tilde{\Sigma}$.*

Under these assumptions the Cauchy problem for equation (1) with initial condition (2) possesses at most one solution in class $\underline{E}_2$ (in class $\bar{E}_2$).)*

The proof of the theorem is based on the following lemma.

Lemma. *Under the assumption that the coefficients of equation (1) are bounded and hypothesis (A) is satisfied, to each solution $u(t, X)$ of equation (1), regular in the zone $\tilde{\Sigma}$ and belonging to class $\underline{E}_2$, there corresponds a solution $v(t, X)$ of the equation*

$$F(v) = -f(t, X),$$

regular and of class E_2 in a certain zone $\approx\Sigma$: $0 \leqq t < T'$ $(T' \leqq T)$ and such that the sum $z(t, X) = u(t, X) + v(t, X)$ is non-negative in $\approx\Sigma$.

The proof of the lemma will be given in a paper which will appear in Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. Ser. VIII, 33, 5.

From the lemma and from theorem 5 it follows that if the solution $u(t, X)$ of equation (1) is of class $\underline{E}_2$, then the function $z(t, X)$ appearing in the lemma is of class I_2, and thus the function $u(t, X)$ is of class I_2. Now in this class the uniqueness of solution of Cauchy problem holds (according to theorem 4). Hence, the validity of theorem 6 results.

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*) Considering the solutions of class $\underline{E}_2$ (of class $\bar{E}_2$) we obviously assume that the function $\psi(X)$ appearing in the initial condition (2) is also of classe $\underline{E}_2$ (class $\bar{E}_2$).

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