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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 30 (1989), No. 2, 83--90

Persistent URL: `http://dml.cz/dmlcz/702154`

1989 ACTA UNIVERSITATIS CAROLINAE—MATHEMATICA ET PHYSICA VOL. 30. NO. 2

An Integral Theorem and its Applications to Coincidence Theorems

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Received 15 March 1989

The purpose of this note is to give an elementary proof of a special case of the Stokes theorem. The theorem presented here is called "Integral Theorem". As an application of the theorem one can obtain an analytical proof of the Brouwer fixed point theorem, a non-retraction theorem and other classical results.

For any natural number $n > 1$ let $P = P(n)$ be the set of all permutations of the natural numbers $1, \ldots, n$ and let $s(\alpha)$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in P$, means the sign of α,

$$s(\alpha) := \prod \{\operatorname{sgn}(\alpha_j - \alpha_i) : i < j, i, j = 1, \ldots, n\} .$$

Assume that $h: U \to \mathbb{R}^n$, $h = (h_1, \ldots, h_n)$, is a differentiable map from an open set $U \subset \mathbb{R}^n$. Recall that the Jacobian $\det h'(x)$ is equal

$$\det h'(x) := \sum_{\alpha \in P} s(\alpha) \left(\frac{\partial h_1}{\partial x_{\alpha_1}} \cdot \ldots \cdot \frac{\partial h_n}{\partial x_{\alpha_n}} \right)(x)$$

Lemma 1. If $h: U \to \mathbb{R}^n$, $n > 1$, is a map of class C^2 from an open set $U \subset \mathbb{R}^n$, then for each point $x \in U$ the following equality holds

$$\det h'(x) = \sum_{\alpha \in P} s(\alpha) \frac{\partial}{\partial x_{\alpha_1}} \left(h_1 \frac{\partial h_2}{\partial x_{\alpha_2}} \cdot \ldots \cdot \frac{\partial h_n}{\partial x_{\alpha_n}} \right)(x) .$$

Proof. Let us calculate

$$\sum_{\alpha \in P} s(\alpha) \frac{\partial}{\partial x_{\alpha_1}} \left(h_1 \frac{\partial h_2}{\partial x_{\alpha_2}} \cdot \ldots \cdot \frac{\partial h_n}{\partial x_{\alpha_n}} \right)(x) = \sum_{\alpha \in P} s(\alpha) \left(\frac{\partial h_1}{\partial x_{\alpha_1}} \cdot \ldots \cdot \frac{\partial h_n}{\partial x_{\alpha_n}} \right)(x) +$$

$$+ \sum_{\alpha \in P} s(\alpha)\, h_1(x) \frac{\partial}{\partial x_{\alpha_1}} \left(\frac{\partial h_2}{\partial x_{\alpha_2}} \cdot \ldots \cdot \frac{\partial h_n}{\partial x_{\alpha_n}} \right)(x) = \det h'(x) + h_1(x)\, r(x) ,$$

where

$$r := \sum_{\alpha \in P} s(\alpha) \frac{\partial}{\partial x_{\alpha_1}} \left(\frac{\partial h_2}{\partial x_{\alpha_2}} \cdot \ldots \cdot \frac{\partial h_n}{\partial x_{\alpha_n}} \right) . \tag{1}$$

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In order to prove the lemma it suffices to show that $r = 0$. Let us note that

$$(2)\qquad \frac{\partial}{\partial x_{\alpha_1}}\left(\frac{\partial h_2}{\partial x_{\alpha_2}}\cdot\ldots\cdot\frac{\partial h_n}{\partial x_{\alpha_n}}\right) = \frac{\partial^2 h_2}{\partial x_{\alpha_1}\,\partial x_{\alpha_2}}\cdot\frac{\partial h_3}{\partial x_{\alpha_3}}\cdot\ldots\cdot\frac{\partial h_n}{\partial x_{\alpha_n}} + \ldots$$

$$\ldots + \frac{\partial h_2}{\partial x_{\alpha_2}}\cdot\ldots\cdot\frac{\partial h_{n-1}}{\partial x_{\alpha_{n-1}}}\cdot\frac{\partial^2 h_n}{\partial x_{\alpha_1}\,\partial x_{\alpha_n}}.$$

Define

$$(3)\qquad H_i^\alpha := \frac{\partial h_2}{\partial x_{\alpha_2}}\cdot\ldots\cdot\frac{\partial h_{i-1}}{\partial x_{\alpha_{i-1}}}\cdot\frac{\partial^2 h_i}{\partial x_{\alpha_1}\,\partial x_{\alpha_i}}\cdot\frac{\partial h_{i+1}}{\partial x_{\alpha_{i+1}}}\cdot\ldots\cdot\frac{\partial h_n}{\partial x_{\alpha_n}}.$$

We have

$$(4)\qquad \frac{\partial}{\partial x_{\alpha_1}}\left(\frac{\partial h_2}{\partial x_{\alpha_2}}\cdot\ldots\cdot\frac{\partial h_n}{\partial x_{\alpha_n}}\right) = \sum_{i=2}^{n} H_i^\alpha$$

and hence

$$(5)\qquad r = \sum_{\alpha\in P} s(\alpha)\sum_{i=2}^{n} H_i^\alpha = \sum_{i=2}^{n}\sum_{\alpha\in P} s(\alpha)\,H_i^\alpha .$$

For each $i = 2, \ldots, n$ let us define a bijection $\lambda_i\colon P - P$,

$$(6)\qquad \lambda_i\alpha = \beta \Leftrightarrow \beta_1 = \alpha_i\,,\quad \beta_i = \alpha_1\ \&\ \beta_j = \alpha_j \quad\text{for}\quad j \neq 1, i\,.$$

From definition of the function s it follows that

$$(7)\qquad s(\lambda_i\alpha) = -s(\alpha)$$

Applying the Schwarz theorem on mixed derivatives:

$$\frac{\partial^2 h}{\partial x_i\,\partial x_j} = \frac{\partial^2 h}{\partial x_j\,\partial x_i},$$

we get

$$(8)\qquad H_i^{\lambda_i\alpha} = H_i^\alpha\,.$$

From (7) and (8) we obtain

$$2\sum_{\alpha\in P} s(\alpha)\,H_i^\alpha = \sum_{\alpha\in P} s(\alpha)\,H_i^\alpha + \sum_{\alpha\in P} s(\lambda_i\alpha)\,H_i^{\lambda_i\alpha} =$$

$$= \sum_{\alpha\in P}\left[s(\alpha) + s(\lambda_i\alpha)\right]H_i^\alpha = \sum_{\alpha\in P}\left[s(\alpha) - s(\alpha)\right]H_i^\alpha = 0\,.$$

In virtue of (5) and from the above we infer that $r = 0$.

Integral Theorem. Let $f, g\colon U \to \mathbb{R}^n$ be maps of class C^2 from an open set $U \subset \mathbb{R}^n$. Assume that $K \subset U$ is a compact set such that $f \mid \mathrm{Bd}\,K = g \mid \mathrm{Bd}\,K$. Then

$$\int_K \det f'(x)\,\mathrm{d}x = \int_K \det g'(x)\,\mathrm{d}x\,.$$

Proof. (I) $n = 1$. Let $K \subset \mathbb{R}$ be a compact set satisfying the assumptions of the Integral Theorem. At first, let us observe that if $x \in \operatorname{Bd} K$ is a non-isolated point in the boundary of the set K then $f'(x) = g'(x)$. Indeed, choose points $x_n \in \operatorname{Bd} K \setminus \{x\}$ such that $x_n \to x$ whenever $n \to \infty$. Since $f(x) = g(x)$ and $f(x_n) = g(x_n)$ for all $n = 1, 2, \dots$, we get

$$f'(x) = \lim_{n\to\infty} \frac{f(x) - f(x_n)}{x - x_n} = \lim_{n\to\infty} \frac{g(x) - g(x_n)}{x - x_n} = g'(x)\,.$$

The interior of the set K is of the form

$$\operatorname{Int} K = \bigcup \{(a_n, b_n)\colon n \in A\}\,,$$

where A is a subset of the set of natural numbers and the intervals (a_n, b_n) are convex components of $\operatorname{Int} K$ (and so they are mutually disjoint). Since K is a compact set, we have $f(a_n) = g(a_n)$ and $f(b_n) = g(b_n)$ for each $n \in A$, and in consequence

$$\int_{a_n}^{b_n} f'(x)\,\mathrm{d}x = \int_{a_n}^{b_n} g'(x)\,\mathrm{d}x$$

and the above implies that

$$\int_K f'(x)\,\mathrm{d}x = \int_K g'(x)\,\mathrm{d}x$$

because the set of all isolated points in $\operatorname{Bd} K$ is at most countable.

(II) $n > 1$. Let $K \subset U$ be a compact subset of an open set $U \subset \mathbb{R}^n$ and assume that $h = (h_1, \dots, h_n)\colon U \to \mathbb{R}^n$ is a map of class C^2. Define

$$I(h_1, \dots, h_n) := \int_K \det h'(x)\,\mathrm{d}x\,. \tag{1}$$

From properties of determinant it follows that

$$I(h_1, \dots, h_n) = s(\alpha)\, I(h_{\alpha_1}, \dots, h_{\alpha_n}) \quad \text{for all} \quad \alpha \in P\,. \tag{2}$$

According to lemma 1 we get

$$I(h_1, \dots, h_n) = \int_K \sum_{\alpha\in P} s(\alpha) \frac{\partial}{\partial x_{\alpha_1}} \left(h_1 \frac{\partial h_2}{\partial x_{\alpha_2}} \cdot \ldots \cdot \frac{\partial h_n}{\partial x_{\alpha_n}} \right)(x)\,\mathrm{d}x\,. \tag{3}$$

Now, we shall verify that if $g_1\colon U \to \mathbb{R}$ is a function of class C^2 such that $h_1 \mid \operatorname{Bd} K = g_1 \mid \operatorname{Bd} K$ then

$$I(h_1, \dots, h_n) = I(g_1, h_2, \dots, h_n)\,. \tag{4}$$

Indeed, let $(x_i, y_i) \in \mathbb{R}_i \times \mathbb{R}^{n-1}$, $\mathbb{R}_i = \mathbb{R}$, means a point $(x_1, \dots, x_i, \dots, x_n) \in \mathbb{R}^n$ and let $\Pi_i\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection; $\Pi_i(x_i, y_i) = y_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots \dots, x_n)$. Define

$$K(y_i) := \{x_i \in \mathbb{R}_i\colon (x_i, y_i) \in K\}\,.$$

Since K is a compact subset of $\mathbb{R}^n$, the set $K(y_i)$ is a compact subset of $\mathbb{R}$, and $h_1(x_i, y_i) = g_1(x_i, y_i)$ for each point $x_i \in \operatorname{Bd} K(y_i)$. According to part (I) the following equality holds for each $\alpha \in P$

$$\begin{aligned}(5)\qquad &\int_{K(y_{\alpha_1})} \frac{\partial}{\partial x_{\alpha_1}}\left(h_1 \frac{\partial h_2}{\partial x_{\alpha_2}} \cdot \ldots \cdot \frac{\partial h_n}{\partial x_{\alpha_n}}\right)(x_{\alpha_1}, y_{\alpha_1})\,\mathrm{d}x_{\alpha_1} = \\ &\int_{K(y_{\alpha_1})} \frac{\partial}{\partial x_{\alpha_1}}\left(g_1 \frac{\partial h_2}{\partial x_{\alpha_2}} \cdot \ldots \cdot \frac{\partial h_n}{\partial x_{\alpha_n}}\right)(x_{\alpha_1}, y_{\alpha_1})\,\mathrm{d}x_{\alpha_1}\,.\end{aligned}$$

From the above and the Fubini theorem we get

$$\begin{aligned} I(h_1, h_2, \ldots, h_n) &= \int_K \sum_{\alpha\in P} s(\alpha) \frac{\partial}{\partial x_{\alpha_1}}\left(h_1 \frac{\partial h_2}{\partial x_{\alpha_2}} \cdot \ldots \cdot \frac{\partial h_n}{\partial x_{\alpha_n}}\right)(x)\,\mathrm{d}x = \\ &= \sum_{\alpha\in P} s(\alpha) \int_{\Pi\alpha_1(K)} \mathrm{d}y_{\alpha_1} \int_{K(y_{\alpha_1})} \frac{\partial}{\partial x_{\alpha_1}}\left(h_1 \frac{\partial h_2}{\partial x_{\alpha_2}} \cdot \ldots \cdot \frac{\partial h_n}{\partial x_{\alpha_n}}\right)(x_{\alpha_1}, y_{\alpha_1})\,\mathrm{d}x_{\alpha_1} = \\ &= \sum_{\alpha\in P} s(\alpha) \int_{\Pi\alpha_1(K)} \mathrm{d}y_{\alpha_1} \int_{K(y_{\alpha_1})} \frac{\partial}{\partial x_{\alpha_1}}\left(g_1 \frac{\partial h_2}{\partial x_{\alpha_2}} \cdot \ldots \cdot \frac{\partial h_n}{\partial x_{\alpha_n}}\right)(x_{\alpha_1}, y_{\alpha_1})\,\mathrm{d}x_{\alpha_1} = \\ &= \sum_{\alpha\in P} s(\alpha) \int_K \frac{\partial}{\partial x_{\alpha_1}}\left(g_1 \frac{\partial h_2}{\partial x_{\alpha_2}} \cdot \ldots \cdot \frac{\partial h_n}{\partial x_{\alpha_n}}\right)(x)\,\mathrm{d}x = I(g_1, h_2, \ldots, h_n)\,.\end{aligned}$$

Now, assuming that $f \mid \mathrm{Bd}\, K = g \mid \mathrm{Bd}\, K$ we get the following sequence of equalities

$$\begin{aligned} &\textstyle\int_K \det f'(x)\,\mathrm{d}x = I(f_1, f_2, \ldots, f_n) \overset{(4)}{=} I(g_1, f_2, \ldots, f_n) \overset{(2)}{=} \\ &- I(f_2, g_1, f_3, \ldots, f_n) \overset{(4)}{=} -I(g_2, g_1, f_3, \ldots, f_n) \overset{(2)}{=} \\ &I(g_1, g_2, f_3, \ldots, f_n) \overset{(2),(4)}{=} \ldots = I(g_1, g_2, \ldots, g_n) = \textstyle\int_K \det g'(x)\,\mathrm{d}x\,.\end{aligned}$$

The proof is completed.

In this part we shall give some applications of the Integral Theorem to fixed point theory. For this purpose let us introduce some definitions and notations.

Let $B := \{x \in \mathbb{R}^n : |x| \leqq 1\}$ be the unit ball and let $S = \mathrm{Bd}\, B$, $S = \{x \in \mathbb{R}^n : |x| = 1\}$ be the boundary of the ball B.

A map $f: B \to B$ is said to be a C^i map, $0 \leqq i \leqq \infty$, if there exists an open set $U \subset \mathbb{R}^n$, $B \subset U$, and a map $\tilde{f}: U \to \mathbb{R}^n$ of class C^i such that $\tilde{f} \mid B = f$.

A map $\varphi: S \to S$ is said to be a C^i_B map, $0 \leqq i \leqq \infty$, if there exists a C^i map $\tilde{\varphi}: B \to B$ such that $\tilde{\varphi} \mid S = \varphi$.

A continuous map $\varphi: S \to S$ has the B-coincidence property if for each pair $f, g: B \to B$ of continuous maps, where $f \mid S = \varphi$, there exists a point $x \in B$ such that $g(x) = f(x)$.

Lemma 2. Let $\varphi: S \to S$ be a C^i_B map for some i, $0 \leqq i \leqq \infty$. Then the following conditions are equivalent:

(i) The exists no C^i map $f: B \to B$ such that $f \mid S = \varphi$ and $f(B) \subset S$.

(ii) The map φ has the B-coincidence property.

(iii) Each continuous map $f: B \to B$ such that $f \mid S = \varphi$, is "onto".

The proof of the lemma is routine. For completness it will be presented here. Let us precede it by some remark on a retraction operation. Describe a map, which each pair $a, b \in B$ of distinct points assigns a unique point $c \in S$ such that the points a, b, c lie on the same line between a and c. It means that there exists a unique number $t \geqq 0$ satisfying the following equation

$$|a + tv| = 1 \quad \text{where} \quad v = \frac{b - a}{|b - a|}.$$

For each two points $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ let xy denotes the scalar product $xy = x_1 y_1 + \ldots + x_n y_n$. We have

$$1 = |a + tv| = aa + 2t\,av + t^2 vv .$$

Since $vv = |v|^2 = 1$ and $|a|^2 = aa$, we get

$$t^2 + 2t\,av - 1 + a^2 = 0$$

$$t = av - \sqrt{[1 - |a|^2 - (av)^2]} .$$

Define

$$R(a, b) := a + tv$$

where

$$t = av - \sqrt{[1 - |a|^2 - (av)^2]} \quad \text{and} \quad v = \frac{b - a}{|b - a|}.$$

From the above it follows that if $f, g \colon U \to \mathbb{R}^n$, $U \subset \mathbb{R}^n$, are continuous maps such that $f(x) \neq g(x)$ for all $x \in U$, then the map

$$r(x) := R(g(x), f(x))$$

is continuous and has the following properties:

(a) If $g(x), f(x) \in B$ then $r(x) \in S$.

(b) If $f(x) \in S$ then $r(x) = f(x)$.

(c) If g, f are of class C^i then r is of class C^i, too, for $i = 0, \ldots, \infty$.

Proof of lemma 2. (i) $\Rightarrow$ (ii). Suppose that $g(x) \neq f(x)$ for all $x \in B$. The compactness of B implies that there exists an $\varepsilon > 0$ such that $|g(x) - f(x)| \geqq \varepsilon$ for all $x \in B$. Put $\delta = \varepsilon/4$. According to the Weierstrass theorem there exists a polynomial $p_1 \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $|g(x) - p_1(x)| < \delta$ for all $x \in B$. Let us put $p(x) := p_1(x)/(1 + \delta)$. Then $p(B) \subset B$ and $|g(x) - p(x)| < 2\delta$ for all $x \in B$, and in consequence $p(x) \neq f(x)$ for all $x \in B$. Now, define $r \colon B \to B$, $r(x) := R(p(x), f(x))$. Properties (a), (b), (c) imply that the map is a C^i map, $\varphi = f \mid S = r \mid S$ and $r(B) \subset S$, contradicting (i).

(ii) $\Rightarrow$ (iii). Suppose that there exists a point $a \in B \setminus f(B)$. Define $s \colon B \to B$, $s(x) := -R(a, f(x))$. According to properties (a), (b), (c) the map $s \colon B \to S$ is continuous and $s(x) = -f(x)$ whenever $f(x) \in S$. Thus $s(x) \neq f(x)$ for all $x \in B$, which is in contradiction with (ii).

(iii) $\Rightarrow$ (i). Obvious.

Recall the following coincidence theorem due to H. Schirmer [5]:

Let $f, g: B \to B$ be continuous maps such that $f(S) \subset S$ and $f \mid S: S \to S$ is not nullhomotopic. Then there exists a point $x \in B$ such that $g(x) = f(x)$.

Observe that the implication (i) $\Rightarrow$ (ii) yields Schirmer's theorem. In our terminology the mentioned theorem means that if $f \mid S$ is not nullhomotopic then $f \mid S$ has the B-coincidence property.

Indeed, if $f \mid S$ has not the B-coincidence property then there exists a continuous map $h: B \to S$ such that $h \mid B = f \mid S$. The map $H: S \times [0, 1] \to S$, $H(x, t) = h(tx)$ accomplishes a homotopy between $f \mid S$ and the constant map $H(x, 0) \equiv h(0)$.

Conversely, each continuous map $H: S \times [0, 1] \to S$ such that $H(x, 1) = f(x)$ and $H(x, 0) \equiv c$ for all $x \in S$, induces a continuous map $h: B \to S$; $h(x) := := H(R(0, x), |x|)$ for $x \in B \setminus \{0\}$ and $h(0) := c$, such that $h \mid S = f \mid S$. Thus we have got the following

Observation. A continuous map $\varphi: S \to S$ has the B-coincidence property if and only if it is not nullhomotopic.

Lemma 3. Assume that $\varphi: S \to S$ is a C^2_B map. If there exists a C^2 map $h: B \to B$, $\varphi = h \mid S$, such that $\det h'(x) \neq 0$ for some point $x \in \operatorname{Int} B$, then φ has the B-coincidence property.

Proof. In view of lemma 2(i) it suffices to show that there is no C^2 map $f: B \to S$, $f = (f_1, \ldots, f_n)$, such that $f \mid S = \varphi$. Suppose that such a map exists. Since $f(B) \subset S$ we have

$$\sum_{i=1}^{n} f_i^2(x) = 1 \quad \text{for all} \quad x \in B$$

so that for all $x \in \operatorname{Int} B$ and $j = 1, \ldots, n$

$$\sum_{i=1}^{n} 2 \frac{\partial f_i}{\partial x_j}(x) f_j(x) = 0 .$$

This implies that

$$\det f'(x) = 0 \quad \text{for all} \quad x \in \operatorname{Int} B ,$$

Another proof one can get immediately from the inverse function theorem which tells us that if $\det f'(x) \neq 0$ for some $x \in \operatorname{Int} B$ then $\operatorname{Int} f(B) \neq \emptyset$.

Now we shall show how to apply the Integral Theorem to discern that a C^2_B map $\varphi: S \to S$ has the B-coincidence property.

Proposition. Let $\varphi: S \to S$ be a C^2_B map. If there exists a C^2 map $h: B \to B$ such that $\varphi = h \mid S$ and

$$\int_B \det h'(x)\, \mathrm{d}x \neq 0$$

then φ has the B-coincidence property.

Proof. Let $f: B \to B$ be a C^2 map such that $f \mid S = h \mid S$. From the Integral Theorem it follows that $\int_B \det f'(x)\, dx \neq 0$. Hence there exists a point $x \in \operatorname{Int} B$ such that $\det f'(x) \neq 0$. Applying lemma 2 we infer that φ has the B-coincidence property.

In the case when $h = \mathit{identity}$ from Proposition we get

Corollary 1. (Bohl-Brouwer fixed point theorem.) Each continuous map $g: B \to B$ has a fixed point.

Corollary 2. (Non-retraction theorem.) Let $K \subset \mathbb{R}^n$ be a compact set. Then each continuous map $m: K \to K$ such that $m(x) = x$ for all $x \in \operatorname{Bd} K$, is "onto".

Proof. Choose an $r > 0$ such that $K \subset B_r$, where $B_r = \{x: |x| \leqq r\}$ and let us define $M: B_r \to B_r$, $M(x) = m(x)$ for $x \in K$ and $M(x) = x$ for $x \in B_r \setminus K$. The map M is continuous and $M \mid \operatorname{Bd} B_r = \mathit{identity}$. Applying Proposition to $h = \mathit{identity}$ we infer that condition (ii) of lemma 1 holds and, in consequence, the map $M: B_r \to B_r$ must be "onto". Hence $m(K) = K$.

Analytical proofs of the non-retraction theorem or the Bohl-Brouwer theorem one can find in [2], [3] and [4].

Another application of Proposition is a new proof of the fundamental theorem of algebra

Every cpmplex non-constant polynomial

$$p(z) = z^m + a_1 z^{m-1} + \ldots + a_m \quad \text{has a zero}\,.$$

Indeed, let us put

$$r := 1 + |a_1| + \ldots + |a_m|$$

$$f(z) := \frac{1}{r^m}(rz)^m = z^m, \quad g(z) := \frac{-1}{r^m}\left[a_1(rz)^{m-1} + \ldots + a_m\right].$$

One can verify that $f(B) \subset B$, $f(S) \subset S$ and $g(B) \subset B$. From the Cauchy-Riemann equations for holomorphic functions we get

$$\det f'(z) = |f'(z)|^2\,.$$

This implies that

$$\int_B \det f'(z)\, dz > 0\,.$$

According to Proposition there exists a point $z \in B$ such that $f(z) = g(z)$. But this is equivalent to $p(z) = 0$.

The above method can be also adopted to prove the fundamental theorems of algebra for quaternions and Cayley numbers. For details the reader is referred to [1].

Concluding the paper I would like to remark that the Integral Theorem can be applied with a success in some cases where in a proof is used the Stokes theorem. Sometimes it leads to a generalization of results because on the contrary to the Stokes theorem it is nothing assumed on the boundary of a compact set K.

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