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ROUGH AND STRONGLY ROUGH NORMS ON BANACH SPACES

BY

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Let X be a real Banach space whose dual is X^*. Their closed unit balls and unit spheres will be denoted B, B^* and S, S^*, respectively. A norm on X is said to be <u>rough</u> (resp., <u>strongly rough</u>), if there is $\varepsilon > 0$ such that for all $x \in X$ and $\eta > 0$ there are $x_1, x_2 \in X$, $u \in S$, (resp., for all $x \in X$ there is $u \in S$ such that for all $\eta > 0$ there are $x_1, x_2 \in X$) such that $\|x_i - x\| < \eta$, $i = 1,2$ and $(d^+\|x_1\| - d^+\|x_2\|)(u) \geq \varepsilon$, where

$$d^+\|x_1\|(u) = \lim_{t\to 0+} \frac{\|x+tu\| - \|x\|}{t} .$$

(This limit exists for all $x \in X$, $u \in S$.)

If X admits an equivalent rough (resp., strongly rough) norm, then there is no real valued Frechet differentiable (resp., continuous Gateaux differentiable) function with bounded nonempty support on X. Also, the existence of an equivalent rough norm on X ensures that there is a separable subspace Y of X with nonseparable dual. [4,5,10].

<u>Theorem 1</u>: The following are equivalent:

(i) $\|\cdot\|$ is not rough.

(ii) B^* is weak* dentable, i.e., for each $\varepsilon > 0$ there is $x \in S$ and $\alpha > 0$ such that $\operatorname{diam}\{f \in B^*: f(x) \geq 1-\alpha\} < \varepsilon$.

(iii) B is strongly smoothable, i.e., for each $\varepsilon > 0$ there are $x \notin B$ and $f \in S^*$ such that $\{x \in B: f(x) \geq \varepsilon\} \subseteq c\ell \bigcup\{t(B-x): t \geq 0\}$.

(iv) $\|\cdot\|$ is malleable, i.e. for each $\varepsilon > 0$ there are $x \in S$ and $\delta > 0$ such that for $0 < \lambda < \delta$ and for any $y \in B$ it follows that $\|x+\lambda y\|+\|x-\lambda y\|-2 < \varepsilon\lambda$.

The equivalence of (ii) and (iv) is essentially due to Sullivan [9]; (ii) equivalent to (iii) is due to Anantharaman, Lewis and Whitfield [1]; and, John and Zizler [4] showed that (i) and (ii) are equivalent. John and Zizler also give the following.

Theorem 2: The following are equivalent:

(i) $\|\cdot\|$ is not strongly rough.

(ii) B^* is weak* weakly dentable, i.e., for every $\varepsilon > 0$ there is $x \in S$ such that $\operatorname{diam}\{f \in B^*: f(x) = 1\} < \varepsilon$.

(iii) $\|\cdot\|$ is weakly malleable, i.e., for each $\varepsilon > 0$ there is $x \in S$ such that for all $y \in B$ there is $\delta > 0$ such that, for $0 < \lambda < \delta$, $\|x+\lambda y\|+\|x-\lambda y\|-2 < \varepsilon\lambda$.

It is easily seen that the dual statement of both theorems obtains.

Problem 1: Is there a geometric condition on B, e.g. similar to strong smoothability, that is equivalent to $\|\cdot\|$ failing strong roughness? Or equivalently, a geometric condition dual to weak dentability?

X is called an Asplund (resp., weak Asplund) space if every continuous convex function on X is Frechet differentiable (resp., Gateaux differentiable) on a dense G_δ subset of its domain. For several properties of such spaces see [6] and [7]. A fairly immediate consequence of Theorem 1 is the following characterization of Asplund spaces.

Theorem 3: (John-Zizler [4]) X is an Asplund space if and only if X does not admit a rough norm.

Some immediate consequences are

Corollary 1: (Namioka-Phelps-Stegall [6,8], see also [1] and [7]) X is an Asplund space if and only if every separable subspaces of X has a separable dual.

Corollary 2: (Leach-Whitfield [5]) If Y is a subspace of X such that dens Y < dens Y*, then X admits an equivalent rough norm.

Corollary 3: (Ekeland-Lebourg [2]) If there is a real valued Frechet differentiable function with bounded nonempty support on X, then X is an Asplund space.

Problem 2: Does the converse of Corollary 3 hold?

A related, but possibly different, problem is:

Problem 3: Does an Asplund space admit an equivalent Frechet differentiable norm?

Less is known about weak Asplund spaces. In our setting we have only

Theorem 4: If X is a weak Asplund space, then X does not admit an equivalent strongly rough norm.

Problem 4: Is the converse of Theorem 4 true?

Problem 5: If X admits an equivalent Gateaux differentiable norm, is X weak Asplund? Converse?

Problem 6: Does the existence of a real valued Gateaux differentiable function with bounded nonempty support on X imply that X is weak Asplund? Converse? Recall that no such function exists if X admits an equivalent strongly rough norm.

X is said to have property (ω) if every bounded sequence in X* has a weak* convergent subsequence.

Theorem 5: (Hagler-Sullivan [3]) If Y is a subspace of X, Y has (ω) and X fails to have (ω), then there is an equivalent strongly rough norm on X/Y. In particular, if X is smooth, then X has (ω).

Also, Stegall [8] has shown that X has (ω) whenever X is weak Asplund. However, the presence of (ω) ensures neither smoothness nor weak Asplund as shown in an example of J. Bourgain. (See [3]).

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