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# On Norm-Attaining Functionals

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This is a short survey on norm-attaining functionals. The first part deals with some basic facts, most known. We streamline the proofs trying to convey some geometrical intuition. The second part focuses on some new facts concerning a problem of Namioka on renormings and norm-attaining functionals.

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## 1. Introduction and notations

Let $(X, \|\cdot\|)$ be a real Banach space, $X^*$ its dual. We shall use the same symbol $\|\cdot\|$ for the the corresponding dual norm in $X^*$ if there is no risk of misunderstand-ing. We shall denote, as usual, $B_{(X,\|\cdot\|)}$ the closed unit ball and $S_{(X,\|\cdot|)}$ the unit sphere of $(X, \|\cdot\|)$. If the norm $\|\cdot\|$ is understood we shall speak about the Banach space $X$. If $S$ is a subset of $X$, $\operatorname{conv}(S)$ stands for its convex hull, $\overline{\operatorname{conv}}(S)$ its closed hull, $\Gamma(S)$ its absolutely convex hull (i.e., the convex hull of the circled hull of $S$) and $\overline{\Gamma}(S)$ its closed absolutely convex hull.

If a subset $K \subset X$ is $w$-compact then it is bounded and every element $x^* \in X^*$ surely attains its supremum on $K$. If $K$ is not $w$-compact, we cannot expect that every element in $X^*$ should attain its supremum on $K$. A fundamental result in weak compactness says that *a closed convex and bounded subset $C$ of $X$ is $w$-compact if and only if every $x^* \in X^*$ attains its supremum on it* [Jai], [Jat]. A number of consequences follow from this important result. For example, the Krein-Milman and the Eberlein-Šmulyan Theorems can be easily deduced from it, as well as the following characterization of reflexivity: *A Banach space $X$ is reflexive iff every element $x^* \in X^*$ attains its supremum on $B_X$.*

Most of the time we shall be interested in the unit ball $B_X$ of $(X, \|\cdot\|)$. The set of elements $x^* \in X^*$ which attain their supremum on $B_X$ (i.e., its norm) is denoted by $A(X)$, or by $A(X, \|\cdot\|)$ if we want to signify the norm on $X$. It is, certainly, a cone in $X$ with vertex at 0. James Theorem for the unit ball of $X$ says then that a Banach space is reflexive if (and only if) $A(X, \|\cdot\|) = X^*$. It is remarkable that *$A(X, \|\cdot\|)$ is always $\|\cdot\|$-dense in $X^*$* [BP]. This result plays also a key role in the Theory (for the failure in the complex case, see [Loq], [Loi] and [Loj]). This survey focuses on the "size" of the sets $A(X, \|\cdot\|)$ and its complement $CA(X, \|\cdot\|) := X^* \setminus A(X, \|\cdot\|)$. The set $A(X, \|\cdot\|)$ depends strongly on the norm, so much that by changing it to an equivalent one it can be completely different, both in shape and in "size". The following problem has been raised several times even in this Winter School (by V. Zizler and V. Montesinos): assume that a Banach space has the property that all $x^* \in X^*$ attain their norm on $B_{(X,\|\cdot|)}$. Let $|||\cdot|||$ be an equivalent norm on $X$. Is the corresponding statement for $(X, |||\cdot|||)$ true, i.e., does every element $x^* \in X^*$ attains $|||x^*|||$ on $B_{(X,|||\cdot|||)}$? Formulated in this way it is not a "problem" at all: we know from James Theorem that then $X$ is reflexive so the answer is, immediately, "yes". The point is to prove the statement by a "simple" geometric argument not depending on James Theorem. This will provide an easier proof of this not-easy-at-all-to-prove result (we may at this moment remark that James provided an accessible proof for the case of separable Banach spaces (see [Ja72]). Simons isolated the combinatorial mechanism in his "Simons inequality", G. Godefroy's "favorite inequality in Analysis" (see, for example, [Gos]). A simple proof of Simons inequality has been given by E. Oja [Oj], and it is recorded in [FHH, Thm. 3.47]). The general result has a very delicate proof (see the paper [Jat], Floret's version in [Fl] and a recent approach by M. Morillon in [Mor]).

An instance of the drastic change in the set $A(X, \|\cdot\|)$ when going to an equivalent norm is worked out below: the set $A(\ell_1, \|\cdot\|_1)$ which has a nonempty $\|\cdot\|_\infty$-interior in $\ell_\infty$ (the norm-interior of a set $M$ will be denoted by $M^0$). However, there is an equivalent norm on $\ell_1$, say $\|\cdot\|$, such that $A^0(\ell_1, \|\cdot\|) = \emptyset$ (see Theorem 23). This was the starting point for providing an (almost) complete solution to Namioka's problem (see 4.2.1 below and the preprint [AM]).

## 2. Some tools

### 2.1 Simons inequality and some of its consequences

Simons inequality is an important tool in the geometry of Banach spaces. It originates in Simon's analysis of James' proof of his weak compactness characterization. It is difficult to give an account of the broad scope of its applications. We refer, for example, to [DGZ, Ch. I.3], [FHH, Ch. 3] and [Gos]. For a proof we refer to [Oj] (see, also, [FHH, Lemma 3.47]). Let's recall the main result. $(\ell_\infty(B), \|\cdot\|_\infty)$ is the space of all real and bounded functions defined on a non-empty set $B$, equipped with the supremum norm $\|\cdot\|_\infty$. Given a bounded set $A$ in a Banach space $X$ we denote by $\operatorname{sconv}(A)$ the set $\{\sum_{n=1}^{\infty}\lambda_n a_n : a_n \in A,\ \lambda_n \geq 0 \text{ for all } n \in \mathbb{N}, \sum_{n=1}^{\infty}\lambda_n = 1\}$ and we call it the *superconvex envelope* of $A$.

**Lemma 1 (Simons inequality).** *Let B be a non-empty set. Let $(x_n)$ be a bounded sequence in $(\ell_\infty(\mathscr{B}), \|\cdot\|_\infty)$ such that every element of* $\operatorname{sconv}\{x_n : n \in \mathbb{N})$ *attains the supremum on B. Let $u :=$* $\limsup x_n$. *Then*

$$\sup u(B) \geq \inf_{x \in \operatorname{conv}\{x_n : n \in \mathbb{N}\}} \sup x(B). \tag{1}$$

A simple consequence is also useful:

**Corollary 2.** *Let X be a Banach space and let A and B be two bounded subsets in $X^*$ such that $B \subset A$. Let $(x_n)$ be a bounded sequence of X and assume that every element of* $\overline{\operatorname{conv}}\{x_n : n \in \mathbb{N}\}$ *attains its supremum on A at some point of B. Let $u :=$* $\limsup x_n$. *Then*

$$\sup u(B) = \sup u(A).$$

**Proof.** Assume $\sup u(B) < \alpha < \sup u(A)$ for some $\alpha$. Find $a \in A$ such that $u(a) > \alpha$. We may and do assume that $x_n(a) > \alpha$ for all $n$. Given $x \in \operatorname{conv}\{x_n : n \in \mathbb{N}\}$ we have $x(a) > \alpha$, hence $\sup x(A) > \alpha$ for all $x \in \operatorname{conv}\{x_n : n \in \mathbb{N}\}$. We get

$$\inf_{x \in \operatorname{conv}\{x_n : n \in \mathbb{N}\}} \sup x(B) \geq \alpha > \sup u(B),$$

a contradiction with (1). □

It is possible to give a simple (and still useful) estimate for the distance (in the corresponding norm in the bidual) from an element $x^{**} \in X^{**}$ to $X$ by computing values of $x^{**}$ on the unit ball of $(X^*, |||\cdot|||)$ for some equivalent norm. Precisely

$$\operatorname{dist}(x^{**}, X) \leq 2 \sup \langle x^{**}, \overline{B}_Y^{w^*} \rangle,$$

where $Y := \operatorname{Ker} x^{**}$. This was obtained from the parallel hyperplane Lemma (see, for example, [FMZ]). In the separable setting another estimate comes from inequality (1); we state and prove it in Lemma 4. Recall that $\partial \|\cdot\| (x) := \{x^* \in B_{X^*} : \langle x, x^* \rangle = \|x\|\}$ for all $x \in X$; this set is called the *subdifferential* of the norm at $x$. The following simple proposition is standard; it will be used in the proof of Lemma 4.

**Proposition 3.** *Let $X$ be a Banach space. Then, given $x, y \in X$, we have $d^+ \|\cdot\| (x)(y) = \sup \langle y, \partial \|\cdot\| (x) \rangle$, where $d^+ \|\cdot\| (x)(y)$ denotes the right derivative of the norm at $x$ in the direction $y$.*

For a proof see, for example, [Phl, Prop. 2.4].

**Remark.** Notice that a consequence of Proposition 3 is that $\|\cdot\|$ in $X^{**}$ is differentiable at some $0 \neq x \in X$ in the direction $x^{**} \in X^{**}$ if and only if $\inf_{\alpha>0} \operatorname{Osc}(x^{**}, S(x, \alpha)) = 0$, where $S(x, \alpha) := \{x^* \in B_{X^*} : \langle x, x^* \rangle > \|x\| - \alpha\}$ and $\operatorname{Osc}(x^{**}, M)$ denotes the oscillation of $x^{**}$ on the set $M \subset X^*$. This is easily seen by observing that $\partial \|\cdot\| (x) = \bigcap_{\alpha>0} \overline{S(x, \alpha)}^{w^*}$ (here $\partial \|\cdot\| (x)$ is the Fréchet differential of $\|\cdot\|$ in $X^{**}$ at $x$, i.e., a subset of $S_{X^{***}}$, and $S(x, \alpha) := \{x^* \in B_{X^*} : \langle x, x^* \rangle > \|x\| - \alpha\}$).

Then we have

**Lemma 4** ([AR1, Lemma 4]). *Let $X$ be a separable Banach space. Assume that there exists $x_0^* \in S_{X^*}$ and some $r > 0$ such that $x_0^* + rB_{X^*} \subset A(X, \|\cdot\|)$. Then, for all $x_0 \in S_X$ such that $\langle x_0, x_0^* \rangle = 1$ we have*

$$\langle x^{**}, x_0^* \rangle + r \operatorname{dist}(x^{**}, X) \leq \sup \langle x^{**}, \partial \|\cdot\| (x_0) \rangle, \tag{2}$$

*where the subdifferential is calculated in $X^{***}$.*

**Proof.** The conclusion is trivial for elements $x^{**} \in X$, so we shall assume $x^{**} \in X^{**} \setminus X$. We shall prove first that

$$\langle x^{**}, x_0^* \rangle + r \operatorname{dist}(x^{**}, X) \leq \|x^{**}\|. \tag{3}$$

Noticing that $d := \operatorname{dist}(x^{**}, X) = \|q(x^{**})\|$, where $y : X^{**} \to X^{**}/X$ is the canonical mapping, we can find $x^{\perp} \in S_{X^{\perp}} \subset X^{***}$ such that $d = \langle x^{**}, x^{\perp} \rangle$. As $X$ is separable we can find a sequence $(x_n^*)$ in $B_{X^*}$ which converges to $x^{\perp}$ on $X \cup \{x^{**}\}$. Fix $\varepsilon > 0$. We may then assume $\langle x^{**}, x_n^* \rangle \geq \langle x^{**}, x^{\perp} \rangle - \varepsilon = d - \varepsilon$ for all $n$. We shall apply Lemma 1 to the sequence $(x_0^* + rx_n^*)_{n=1}^{\infty}$ and the set

$B := B_X$. From now on we work in $\ell_\infty(B)$. Obviously, $u := \limsup(x_0^* + rx_n^*) = x_0^*$. From (1) we get

$$1 = \sup x_0^*(B) \geq \inf_{x^* \in \operatorname{conv}\{x_0^* + rx_n^*\}} \sup x^*(B) = \inf_{x^* \in \operatorname{conv}\{x_0^* + rx_n^*\}} \|x^*\|. \qquad (4)$$

We also have, for all $n$,

$$\left\langle \frac{x^{**}}{\|x^{**}\|}, x_0^* + rx_n^* \right\rangle \geq \frac{1}{\|x^{**}\|}\left(\langle x^{**}, x_0^* \rangle + r(d - \varepsilon)\right).$$

This applies also to every element $x^* \in \operatorname{conv}\{x_0^* + rx_n^*\}$, so

$$\|x^*\| \geq \frac{1}{\|x^{**}\|}\left(\langle x^{**}, x_0^* \rangle + r(d - \varepsilon)\right)$$

and we obtain from (4)

$$1 \geq \frac{1}{\|x^{**}\|}\left(\langle x^{**}, x_0^* \rangle + r(d - \varepsilon)\right),$$

from what (3) follows letting $\varepsilon$ tend to 0.

In order to prove (2), fix $x_0 \in S_X$ such that $\langle x_0, x_0^* \rangle = 1$. For $t > 0$ apply (3) to $x_0 + tx^{**}$ to obtain

$$\langle x_0 + tx^{**}, x_0^* \rangle + r \operatorname{dist}(x_0 + tx^{**}, X) \leq \|x_0 + tx^{**}\|,$$

i.e.,

$$1 + t\langle x^{**}, x_0^* \rangle + rtd \leq \|x_0 + tx^{**}\|.$$

We get

$$\langle x^{**}, x_0^* \rangle + rd \leq \frac{\|x_0 + tx^{**}\| - \|x_0\|}{t}.$$

Letting $t \downarrow 0$ we get

$$\langle x^{**}, x_0^* \rangle + rd \leq d^+ \|\cdot\|(x_0)(x^{**}).$$

In order to obtain (2) it is enough now to apply Proposition 2. □

**Remark.** Some facts follow from the estimates (2) and (1). For example

1. If $x^{**}$ attains the supremum on $B_{X^*}$ at some norm-interior element of $A(X, \|\cdot\|)$, then $x^{**} \in X$. In other words, the face of $B_{X^{**}}$ defined by some element $x^* \in A^0(X, \|\cdot\|)$ is contained in $X$. This ensures reflexivity of a separable $X$ such that $A(X, \|\cdot\|) = X^*$, proving James' Theorem in the separable case.

2. *Let $(X, \|\cdot\|)$ be a Banach space such that $A(X, \|\cdot\|)$ has non-empty interior (in norm). Then, every sequence $(x_n^*)$ in a convex $\|\cdot\|$-open subset $O$ of $A(X, \|\cdot\|)$ which is $w^*$-null is $w$-null.* In order to prove it, assume, on the contrary, that $(x_n^*)$ is not $w$-null. Then, by passing to a subsequence if

necessary, we can find $x^{**} \in S_{X^{**}}$ and some $\varepsilon > 0$ such that $\langle x^{**}, x_n^* \rangle > \varepsilon$ for all $n \in \mathbb{N}$. From now on, everything will be done in $\ell^\infty(B_X)$. The set $O$ is $\|\cdot\|$-open and convex, hence superconvex, and it is contained in $A(X, \|\cdot\|)$. Put $u := \limsup x_n^*$. This is just 0. By Simons inequality (1) we have

$$0 \geq \inf_{x^* \in \operatorname{conv}\{x_n^*\}_{n=1}^{+\infty}} \sup x^*(B_X). \tag{5}$$

Notice that for $x^* \in \operatorname{conv}\{x_n^*\}_{n=1}^{+\infty}$ we have $\langle x^{**}, x^* \rangle \geq \varepsilon$. It follows that $\sup x^*(B_X) (= \sup x^*(B_{X^{**}})) \geq \varepsilon$, and we reach a contradiction with (5).

## 3. When $A(X, \|\cdot\|)$ is small

Take the separable Banach space $(c_0, \|\cdot\|_\infty)$. It is very easy to show that $A(c_0, \|\cdot\|_\infty)$ is the subspace $\varphi \subset \ell_1$ of all elements in $\ell_1$ with finite support. This set is the union of a countable number of finite-dimensional subspaces of $\ell_1$, so it is 1st category (in particular $CA(c_0, \|\cdot\|_\infty)$ is dense and $A(c_0, \|\cdot\|_\infty)$ has empty interior) in $(\ell_1 \|\cdot\|_1)$.

The closed unit ball of $(c_0, \|\cdot\|_\infty)$ is not dentable, i.e., we cannot produce sections of arbitrary small diameter. In fact, every section of $B_{(c_0, \|\cdot\|_\infty)}$ has norm-diameter 2. It is not by chance that we chose $c_0$ as an example of how small the set $A(X, \|\cdot\|)$ can be: in fact, all separable Banach spaces sharing this property with $(c_0, \|\cdot\|_\infty)$ (i.e., having non-dentable closed unit balls) behave in the same way: the set $A(X, \|\cdot\|)$ is 1st category in $(X^*, \|\cdot\|)$. This result is due to Bourgain and Stegall (see, for example, [Bou, Thm. 3.5.5]). The proof needs two very useful geometric lemmata. The first one, due to Phelps, has been called *the parallel hyperplane lemma:*

**Lemma 5 (Parallel-hyperplane).** *Let $X$ be a Banach space and $x^*$ and $y^*$ two norm-one elements in $X^*$ such that, for some $\delta \geq 0$ we have $|\langle x, y^* \rangle| \leq \delta$ for every $x \in B_X \cap \operatorname{Ker} x^*$. Then $\|x^* - y^*\| \leq 2\delta$ or $\|x^* + y^*\| \leq 2\delta$.*

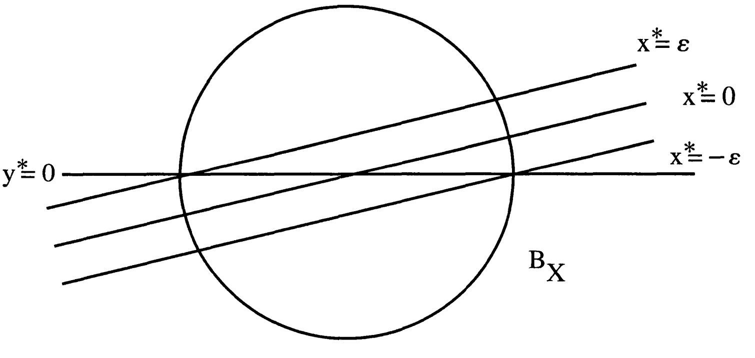


Figure 1: The Parallel Hyperplane Lemma

For a simple proof we refer to [FHH, Ex. 3.1]. This result has been recently extended to the setting of multilinear forms [ACGZ].

The second one is due to Bourgain and Namioka and is frequently referred as the *Superlemma.* We provide a version of the statement equivalent, as it is easy to prove, to the original one (see, for example, [Dies]), although easier to visualize:

**Lemma 6 (Superlemma).** *Let $X$ be a Banach space. Let $A$ and $B$ be two closed convex and bounded subsets of $X$ such that $B \not\subset A$ and* $\operatorname{diam}(B) < \varepsilon$ *for some* $\varepsilon > 0$. *Thes there exists a slice $S$ of $\overline{\operatorname{conv}}(A \cup B)$ intersecting $B$ and with* $\operatorname{diam}(S) < \varepsilon$.

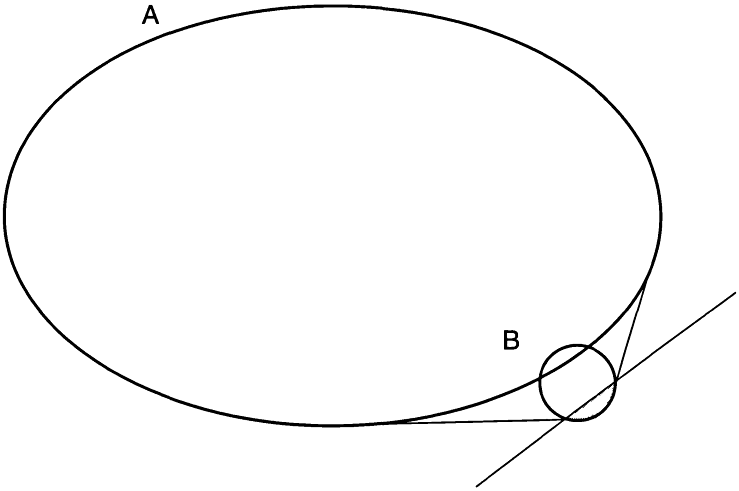


Figure 2: The Superlemma

**Theorem 7 (Bourgain, Stegall,** see [Bou, Thm. 3.5.5]). *Let $K$ be a separable closed, bounded, convex and non-dentable subset of a Banach space $X$. Then, the set of all elements in $X^*$ attaining their suprema on $K$ is $1^{st}$ category in $(X^*, \|\cdot\|)$.*

**Proof.** There exists some $\delta > 0$ such that every slice of $K$ has a diameter $\geq 3d$. Let $\{x_n : n \in \mathbb{N}\}$ be a dense subset of $K$. Define, for $n \in \mathbb{N}$,

$$O_n := \{x^* \in X^* : x^* \text{ defines a slice of } K \text{ disjoint from } B(x_n; \delta) \cap K\}.$$

Obviously, $O_n$ is norm-open for every $n \in \mathbb{N}$, and $A(K, \|\cdot\|) \subset X^* \setminus \bigcap_{n=1}^{\infty} O_n$. It is enough to prove that, for every $n \in \mathbb{N}$, $O_n$ is norm-dense. To that end, fix $n \in \mathbb{N}$ and $x^* \in X^*$ and let $0 < \varepsilon < 1$ be arbitrary. We shall prove that there exists $y^* \in O_n$ such that $\|x^* - y^*\| < \varepsilon$. Observe first that $O_n$ is closed under multiples by positive scalars, so we may henceforth assume $\|x^*\| = 1$. Pick $y \in X$ such that $\langle x - y, x^* \rangle > 0$ for $x \in K$ and let $M := \sup\{\|x - y\| : x \in K\}$. Choose $t \geq 2M/\varepsilon$, let $V := \operatorname{Ker} x^* \cap B(0; t)$ and $C := V + y$. Apply Lemma 6 to the sets

$B(x_n;\delta) \cap K$ and $C$ to obtain a slice of $\overline{\text{conv}}\{(B(x_n;\delta) \cap K) \cup C\}$ intersecting $B(x_n;\delta) \cap K$ and having diameter $< 3\delta$. If $K \subset \overline{\text{conv}}\{(B(x_n;\delta) \cap K) \cup C\}$ we should obtain a slice of $K$ of diameter $< 3\delta$, a contradiction. It follows that there exists $x_0 \in K \setminus \overline{\text{conv}}\{(B(x_n;\delta) \cap K) \cup C)$. It is enough now to separate $x_0$ and $\overline{\text{conv}}\{(B(x_n;\delta) \cap K) \cup C\}$ using some $y^* \in S_{X^*}$; this element obviously belongs to $O_n$ and Lemma 5 gives $\|x^* - y^*\| \leq \varepsilon$.

**Remark.** To our knowledge, it is not known whether Theorem 7 holds without the separability assumption.

Another result in the same vein was obtained by Talagrand (see [Bou, p. 58]) and Kenderov, Moors and Sciffer [KMS]:

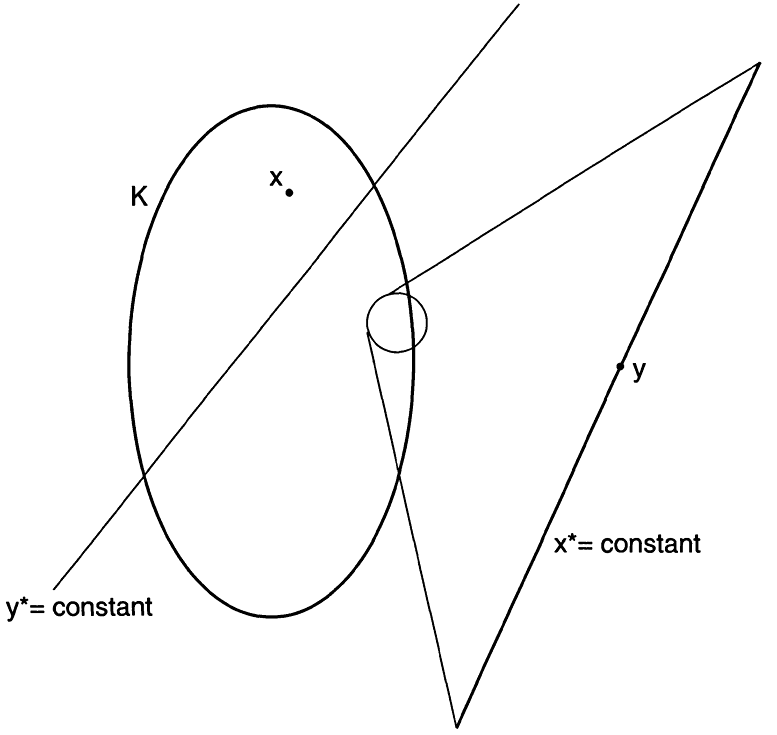


Figure 3: Prof of Theorem 7

**Theorem 8 (Talagrand, Kenderov, Moors, Sciffer).** *Let $K$ be an infinite compact topological space. Then $A(C(K), \|\cdot\|_\infty)$ is $1^{st}$ category in $C(K)^*$.*

There is a kind of converse. Recall that a subset $M$ of a Banach space $X$ has the Radon-Nikodým Property (RNP, in short) if every closed, bounded and convex subset of $M$ is dentable.

**Theorem 9** ([Bou, Cor. 3.5.7]). *Let C be a closed convex subset of a Banach space X. Then, the following are equivalent:*

1. *C has RNP.*
2. *Every (non empty) closed convex and bounded subset K of C satisfies*

$$K = \overline{\text{conv}}\,(str.\,exp\,(K)),$$

*where str. exp $(K)$ denotes the set of strongly exposed points of K.*

3. *Every (non empty) closed convex and bounded subset K of C satisfies that the set of elements in $X^*$ attaining their suprema on K is $2^{nd}$ category.*

**Remark.** Obviously, if $A(X, \|\cdot\|)$ is $1^{st}$ category then $CA(X, \|\cdot\|)$ is dense and $A(X, \|\cdot\|)$ has an empty norm-interior.

## 4. When $A(X, \|\cdot\|)$ is big

### 4.1 The isometric approach

#### 4.1.1 Using the norm topology

We fix a norm on a Banach space $X$. As every metric concept used here refers to this norm, we will omit $\|\cdot\|$ from the subsequent expressions. So, for example, $A(X)$ will denote the set of elements in $X^*$ attaining their norms on the closed unit ball $B_X$ of $X$.

In some sense, the set $A(X, \|\cdot\|)$ is always big. This is the content of the Bishop-Phelps Theorem:

**Theorem 10 (Bishop-Phelps** [BP]**).** *For every Banach space $(X, \|\cdot\|)$, the set $A(X, \|\cdot\|)$ is always norm-dense in $X^*$.*

If $A(X)$ is as large as possible, the space is already reflexive. This is James Theorem:

**Theorem 11 (James** [Jaa]**).** *Let X be a Banach space. Then, X is reflexive if (and only if) $A(X) = X^*$.*

Let us try to obtain a similar result by diminishing the requirements. We can state that *$X$ is reflexive if* 0 *belongs to the norm-interior of $A(X)$*. However, this is not an improvement: $A(X)$ is a cone. As soon as 0 belongs to its norm-interior we have $A(X) = X^*$ and we are in James Theorem' setting.

A naive conjecture is that $A(X)$ having a non-empty norm-interior will be sufficient for the reflexivity of $X$. Maybe a sort of translation will force then 0 to be a norm-interior point of $A(X)$. That this (dramatically) fails can be seen by the (very easy to prove) fact that every Banach space has an equivalent norm such that, in this new norm, $A(X)$ has a non-empty norm interior (see [AR1]). Right now we can state the following result.

**Proposition 12.** *Let $X$ be a Banach space such that $B_X = \overline{\Gamma}(E)$ for some subset $E \subset X$. Assume that there exist $e_0^* \in S_{X^*}$ and $e_0 \in E$ such that $\langle e_0, e_0^* \rangle > \sup \{\langle e, e_0^* \rangle : e \in E \setminus \{e_0\}\}$ (such a point $e_0$ is called a* strong vertex*). Then, $e_0^*$ is in the norm-interior of $A(X)$.*

**Proof.** It is obvious that if $\|x^* - e_0^*\|$ is small enough, then $x^*$ still attains its norm (at the same point $e_0$). □

**Remark.** $(\ell_1, \|\cdot\|_1)$ has this property (and is not reflexive). We shall see later (see Proposition 20) that every Banach space can be renormed to satisfy the condition in Proposition 12.

It is remarkable that a special kind of norm-open subset of the set $A(X, \|\cdot\|)$ forces the space to be reflexive.

**Proposition 13 (Jiménez-Sevilla, Moreno** [JiM]**).** *Let $(X, \|\cdot\|)$ be a Banach space. Assume that $A(X, \|\cdot\|)$ contains a section $S(x^{**}; \delta) := \{x^* \in B_{X^*}; \langle x^{**}, x^* \rangle > 1 - \delta\}$ for some $x^{**} \in S_{X^{**}}$ and $\delta > 0$. Then $X$ is reflexive.*

The key idea of the proof of this result is to use in the dual the (dual) norm whose unit ball is given by

$$B = (S(x^{**}; \delta) - x_0^*) \cap (- S(x^{**}; \delta) + x_0^*),$$

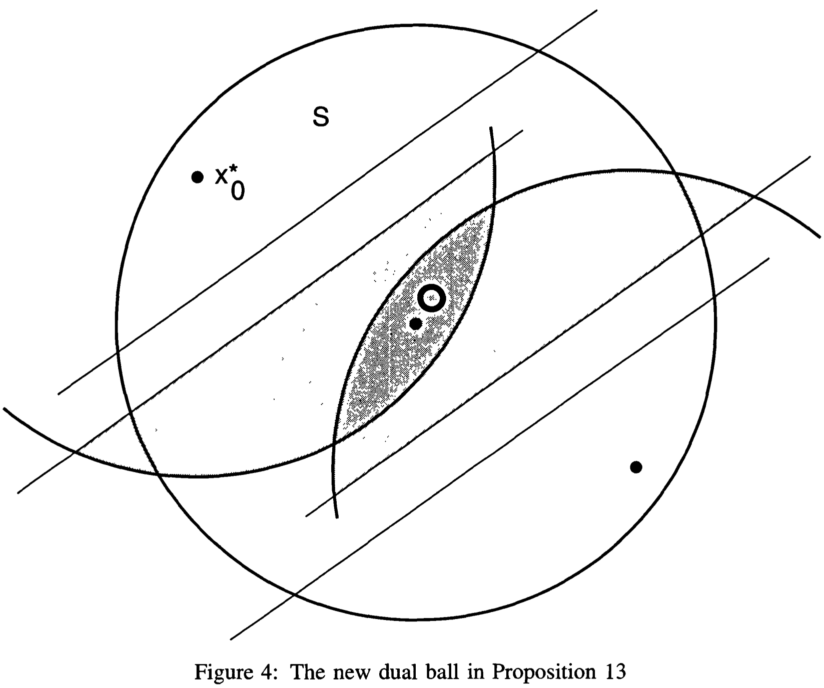


Figure 4: The new dual ball in Proposition 13

where the element $x_0^* \in X^*$ satisfies that $\| x_0^*\| < 1$ and $x^{**}(x_0^*) > 1 - \frac{\delta}{2}$ and then apply James' Theorem.

It is natural now to look at the denting points of the unit ball of the dual in order to obtain characterizations of reflexivity in terms of the set of norm-attaining functionals. This is why we focus on two special properties. Recall first that a point $x \in S_{(X,\|\cdot\|)}$ is *denting* if for every $\varepsilon > 0$ there exists $x^* \in S_{(X^*,\|\cdot\|)}$ defining a slice of $B_{(X,\|\cdot\|)}$ which contains $x$ and has diameter $< \varepsilon$. Similarly, a point $x^* \in S_{(X^*,\|\cdot\|)}$ is *$w^*$-denting* if for every $\varepsilon > 0$ there exists $x \in S_{(X,\|\cdot\|)}$ defining a slice of $B_{(X^*,\|\cdot\|)}$ which contains $x^*$ and has diameter $< \varepsilon$.

**Definition 14.** *A Banach space $(X, \|\cdot\|)$ has the* Mazur Intersection Property *(in short, MIP) whenever every closed convex and bounded subset of $X$ is an intersection of closed balls. A dual Banach space $X^*$ has the* $w^*$-Mazur Intersection Property *(in short, $w^*$-MIP) whenever every convex and $w^*$-compact subset is an intersection of closed balls.*

MIP has been characterized by Giles, Gregory and Sims [GGS] in the following terms: *a Banach space $X, \|\cdot\|)$ has MIP if the only if the set of $w^*$-denting points of $B_{(X^*,\|\cdot\|)}$ is dense is $S_{(X^*,\|\cdot\|)}$*. Similarly, *a dual Banach space $(X^*, \|\cdot\|)$ has $w^*$-MIP if and only if the set of denting points of $B_{(X,\|\cdot\|)}$ is dense in $S_{(X,\|\cdot\|)}$* [GGS]. With this characterizations at hand, it is now obvious that if $(X, \|\cdot\|)$ has MIP then $(X^{**}, \|\cdot\|)$ has $w^*$-MIP. It is also clear (use the Bishop-Phelps Theorem and the Šmulyan characterization of Fréchet differentiability) that *every Banach space with a Fréchet differentiable norm has MIP* [Ph].

We have the following renorming characterization of reflexivity:

**Theorem 15 (Jiménez-Sevilla, Moreno** [JiM]**).** *Let $(X, \|\cdot\|)$ be a Banach space. Then the following are equivalent:*

*(i) $X$ is reflexive.*

*(ii) There exists in $X$ an equivalent norm $|\|\cdot\||$ with MIP and such that $A(X, |\|\cdot\||)$ has non-empty norm-interior.*

*(iii) There exists in $X$ an equivalent norm $|\|\cdot\||$ such that $(X^{**}, |\|\cdot\||)$ has $w^*$-MIP $A(X, |\|\cdot\||)$ has non-empty norm-interior.*

**Proof.** *(i) ⇒ (ii)* follows from the classical Troyanski's renorming theorem: there exists an equivalent dual LUR norm $|\|\cdot\||$ in $X^*$, so the corresponding norm $|\|\cdot\||$ in $X$ is Fréchet differentiable; hence $(X, |\|\cdot\||)$ has MIP (and it is reflexive, so $A(X, |\|\cdot\||) = X^*$). *(ii) ⇒ (iii)* is obvious. *(iii) ⇒ (i)*. If $(X^{**}, |\|\cdot\||)$ has $w^*$-MIP then the set of denting points of $S_{(X^*,|\|\cdot\||)}$ is dense. At least one of them is in the norm-interior of $A(X, |\|\cdot\||)$, hence there exists a slice of $B_{X^*,|\|\cdot\||)}$ inside $A(X, |\|\cdot\||)$. It is enough to apply now Proposition 13. □

**Definition 16.** *A Banach space $(X, \|\cdot|)$ is* very smooth *if the duality mapping $x \to \partial \|\cdot\|(x)$ from $X \setminus \{0\}$ into $S_{X^*}$ is $\|\cdot|$-$w$-continuous.*

This definition is due to Diestel and Faires [DF]. They proved, for example, that *if $X^{**}$ is very smooth then the space $X$ is reflexive* (see, for example, [Dieg]).

The following result gives, in the separable case, another characterization of reflexivity in terms of the set of norm-attaining functionals, now using the property of being very smooth.

**Theorem 17 (Acosta, Ruiz-Galán** [AR1]**).** *Let $(X, \|\cdot\|)$ be a separable Banach space. Then the following are equivalent:*

*(i) $X$ is reflexive.*
*(ii) There exists an equivalent norm $|||\cdot|||$ in $X$ such that $(X, |||\cdot|||)$ is very smooth and $A(X, |||\cdot|||)$ has a non-empty norm-interior.*

**Proof.** *(i) ⇒ (ii)* is obvious. *(ii) ⇒ (i)*: Let $x_0^* \in A^0(X, \|\cdot\|)$, $\|x_0^*\| = 1$. Let $x_0 \in S_{(X,\|\cdot\|)}$ be such that $\langle x_0, x_0^* \rangle = 1$. If $(X, \|\cdot\|)$ is very smooth, $\partial \|\cdot\| (x_0) \subset S_{(X^{***},\|\cdot\|)}$ consists of only one point, namely $x_0^*$. Then use inequality (2) to conclude that $x^{**} \in X$ for every $x^{**} \in X^{**}$, so $X$ is reflexive. □

By [AR2, Thm. 1], the previous characterization also holds in general.

### 4.1.2 Using some other topologies

The use of the $w^*$ topology in the evaluation of "how big" the set $A(X, \|\cdot\|)$ is gives way to some other sufficient conditions for reflexivity. Debs, Godefroy and Saint-Raymond produced the following result. We shall omit its proof as it was exteded by Jiménez-Sevilla and Moreno to non-necessarily-separable Banach spaces.

**Theorem 18 (Debs, Godefroy, Saint-Raymond** [DGS]**).** *Let $(X, \|\cdot\|)$ be a separable Banach space such that $A(X, \|\cdot\|)$ has a non-empty $w^*$-interior. Then $X$ is reflexive.*

**Theorem 19 (Jiménez-Sevilla, Moreno** [JiM]**).** *Given a Banach space $X$, either the set $C := S_{(X^*,\|\cdot\|)} \setminus A(X, \|\cdot\|)$ is $w^*$-dense in $S_{X^*,\|\cdot\|)}$ or $X$ is reflexive. Also, in the first case, $\overline{\text{conv}}^{\|\cdot\|}(C) = B_{(X^*,\|\cdot\|)}$.*

In order to prove the first assertion, it is assumed that $C$ is not $w^*$-dense in the unit sphere of the dual. Then the key idea is to use the dual norm whose unit ball is given by

$$U := (W - y_0^*) \cap (y_0^* - W),$$

where $W$ is a $w^*$-closed set given by an intersection of slices of the dual unit ball and $y_0^*$ is a certain element in the norm interior of $W$. Then it is shown that for this new norm the assumption of James' Theorem is satisfied. For the second part, if $\overline{\text{conv}}^{\|\cdot\|}(C)$ were a proper set of the dual unit ball, then Proposition 13 can be applied and $X$ will be reflexive, a contradiction.

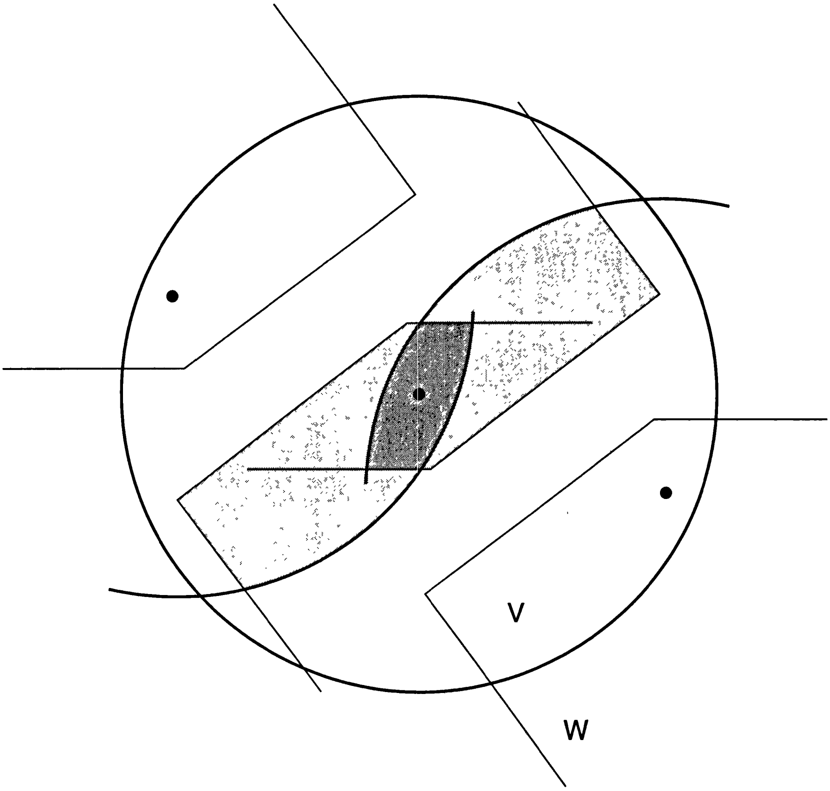


Figure 5: The new dual ball in Theorem 19

### 4.2 The isomorphic approach

We can trace the origin of James' characterization of reflexivity in an earlier work by Klee: he proved [Kl] that *a Banach space $X$, with the property that for each equivalent norm $|||\cdot|||$ we have $A(X,|||\cdot|||) = X^*$, is reflexive.* Of course, this was superseded by the theorem of James. However, we shall return to Klee's original approach (i.e., allowing equivalent norms) asking less in order to obtain reflexivity.

First of all, it is an elementary (and known) fact that every Banach space can be renormed to have a set of norm-attaining functionals with a non-empty interior. This is made precise in the following

**Proposition 20.** *Let $(X,\|\cdot\|)$ be a Banach space. Take $e_0 \in X \setminus B_{(X,\|\cdot\|)}$. Then there exists an equivalent norm $|||\cdot|||$ in $X$ such that $e_0 \in S_{(X,|||\cdot|||)}$ is a strong vertex of $B_{(X,|||\cdot|||)}$. In particular, the set $A(X,|||\cdot|||)$ has a non-empty norm-interior.*

**Proof.** Define the ball of the new norm $|||\cdot|||$ as $\operatorname{conv}\{B_{(X,\|\cdot\|)} \cup \{\pm e_0\}\}$. It is obvious then that $e_0$ is a strong vertex of the new norm (take a functional $x^* \in X^*$ strongly separating $\{e_0\}$ and $B_{(X,\|\cdot\|)}$; then $x^*$ attains the supremum on $B_{(X,|\|\cdot\||}$ precisely at $e_0$). Apply now Proposition 12. □

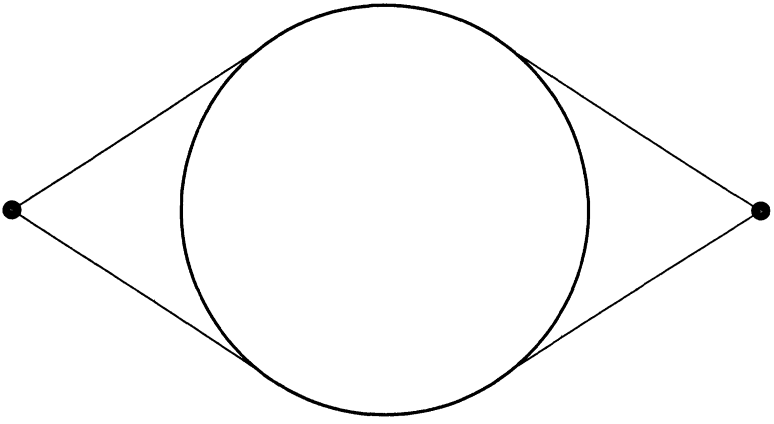

Figure 6: The new unit ball in Proposition 20

It is plain now that, in a very dramatic way, the conjecture that having a non-empty norm-open set of norm-attaining functionals is not enough for reflexivity.

### 4.2.1 A conjecture of I. Namioka

I. Namioka, in a seminar hold in Murcia (Spain) in 1999, posed the following problem:

***Assume $(X, \|\cdot\|)$ is a Banach space such that for every equivalent norm $|\|\cdot\||$ the set $A(X, |\|\cdot\||)$ has a non-empty norm interior. Is then $X$ reflexive?***

No answer was known even for separable Banach spaces.

There were early attempts to solve this problem. A partial answer was given in the following.

**Theorem 21 (Acosta, Ruiz-Galán** [AR1]**).** *Let $(X, \|\cdot\|)$ be a separable Banach space. If $X$ is not weakly seqentially complete then there exists an equivalent norm $|\|\cdot\||$ such that $A^0(X, |\|\cdot\||) = \emptyset$.*

The theorem relies heavily on a renorming result of Godefroy. Precisely, when studying rough norms in Banach spaces, he proved the following result (part of a broader one, see [Gom, Thm. I.2]):

**Theorem 22 (Godefroy,** [Gom]**).** *Let $(X, \|\cdot\|)$ be a separable Banach space. Let $x^{**} \in X^{**} \setminus X$ such that there exists a sequence $(x_n)$ in $X$ which w*-converges* *to $x^{**}$. Then there exists an equivalent norm $|||\cdot|||$ on $X$ such that the corresponding norm $|||\cdot|||$ on $X^{**}$ is differentiable in the direction $x^{**}$ at every point $x \in X$.*

**Proof.** Let $(d_n)$ be a dense sequence in $B_X$. The norm

$$\|x^*\|_1 := \|x^*\| + \left(\sum_{k=1}^{\infty} 2^{-k}\langle d_k, x^*\rangle^2\right)^{1/2}, \quad x^* \in X^*, \tag{6}$$

is an equivalent strictly convex dual norm in $X^*$. We define now a second equivalent dual norm $|||\cdot|||$ on $X^*$ by the formula

$$|||x^*||| := \|x^*\|_1 + \sum_{n=1}^{\infty} 2^{-n} \sup_{k,l\geq n} |\langle x_k - x_l, x^*\rangle|. \tag{7}$$

The norm $\|\cdot\|_1$ is strictly convex. It follows that $|||\cdot|||$ is strictly convex, too. Hence $|||\cdot|||$ is the dual norm of a smooth norm at $S_X$. If we can show that

$$|||x_i^*||| \leq 1, \quad |||x^*||| = 1, \quad x_i^* \xrightarrow{w^*} x^* \tag{8}$$

implies $\langle x^{**},x_i^*\rangle \to \langle x^{**},x^*\rangle$ then, according to Proposition 3 and the remark after it, we are done. Since the functionals defining $|||\cdot|||$ are $w^*$ – lsc, the conditions (8) imply that

$$\text{for every } n \in \mathbb{N}, \ \lim_i \sup_{k,l\geq n} |\langle x_k - x_l, x_i^*\rangle| = \sup_{k,l\geq n} |\langle x_k - x_l, x^*\rangle|. \tag{9}$$

Take $\varepsilon > 0$. Since $(x_n)$ converges to $x^{**}$ in the $w^{**}$ topology, then it is a $w$-Cauchy sequence, so there is $N$ such that

$$\sup_{l\geq N} |\langle x^{**} - x_l, x^*\rangle| < \varepsilon, \tag{10}$$

and also

$$\sup_{k,l\geq N} |\langle x_k - x_l, x^*\rangle| < \varepsilon. \tag{11}$$

By the $w^*$-convergence of $(x_n^*)$ to $x^*$,

$$\exists I_1, i \geq I_1, \quad |\langle x_N, x_i^* - x^*\rangle| \leq \varepsilon. \tag{12}$$

From (11) and (9), there is $I_2 \geq I_1$ such that

$$i \geq I_2 \Rightarrow \sup_{k,l\geq N} |\langle x_k - x_l, x_i^*\rangle| < \varepsilon,$$

and so

$$i \geq I_2 \Rightarrow \sup_{l\geq N} |\langle x^{**} - x_l, x_i^*\rangle| \leq \varepsilon, \tag{13}$$

By using (10), (12) and (13) we deduce that for $i \geq I_2$ it holds

$$|\langle x^{**},x^* - x_i^*\rangle| < 3\varepsilon,$$

as we wanted to show. □

**Proof of Theorem 21.** If $X$ is not weakly sequantially complete, we can find an element $x^{**} \in X^{**} \setminus X$ which is the $w^*$-limit of a sequence in $X$. From Theorem 22 there exists an equivalent norm $|||\cdot|||$ on $X$ such that the corresponding norm $|||\cdot|||$ on $X^{**}$ is diffentiable at every point $x \in X$ in the direction $x^{**}$. Assume that $A(X, |||\cdot|||)$ contains a norm-interior point $x_0^*$. Since the set of norm attaining functionals is a cone, we can assume that $x_0^* \in S_{(X^*,|||\cdot|||)}$. Choose $x_0 \in S_{(X,|||\cdot|||)}$ such that $\langle x_0, x_0^* \rangle = 1$. Observe now that $\langle x^{**}, x^{***} \rangle = \langle x^{**}, x_0^* \rangle$ for all $x^{***} \in \partial |||\cdot|||(x_0) \subset S_{(X^{***},|||\cdot|||)}$. It is enough to use (2) to get $\mathrm{dist}_{|||\cdot|||}(x^{**}, X) = 0$, a contradiction. □

In view of Theorem 21 it is natural to ask what is the situation in the nonreflexive weakly sequentially complete case. The prototype of such a space is $\ell_1$. The following result holds:

**Theorem 23.** ([AM, Thm. 1]) *There is an equivalent norm* $\|\cdot\|$ *in* $(\ell_1, \|\cdot\|_1)$ *such that* $A^0(\ell_1, \|\cdot\|) = \emptyset$.

In fact, the set $B = B_{\ell_1} + A$, where $B_{\ell_1}$ is the closed unit ball for the usual norm in $\ell_1$ and

$$A = \overline{\mathrm{conv}}^{w^*}\left\{\sum_{n=1}^{\infty} \lambda_n \frac{1}{2^n} e_{\sigma(n)} : |\lambda_n| = 1,\ \forall n,\ \sigma : \mathbb{N} \to \mathbb{N} \text{ injective}\right\}$$

is the unit ball of a norm satisfying the previous condition.

The above result can be used in order to get the following extension:

**Theorem 24.** ([AM, Thm. 2]) *Let* $(X, \|\cdot\|)$ *be a Banach space with an isomorphic copy of* $\ell_1$. *Then* $X$ *has an equivalent norm* $p$ *such that* $A^0(X, p) \neq \emptyset$.

**Proof.** Consider $\ell_1$ as a vector subspace of $X$, so $\|\cdot\|$ induces on $\ell_1$ a norm (again denoted by $\|\cdot\|$) which is equivalent to $\|\cdot\|_1$, the canonical norm of $\ell_1$. Then $(\ell_1, \|\cdot\|)^* = (\ell_\infty, \|\cdot\|)$, where $\|\cdot\|$ denotes also the dual norm both on $X^*$ and on $\ell_\infty$ (in the last case, a norm equivalent to $\|\cdot\|_\infty$. Denote by $q : X^* \to \ell_\infty$ the quotient mapping. Now, $c_0 \subset \ell_\infty$ is a norming subspace for $(\ell_1, \|\cdot\|)$ (not necessarily 1-norming), hence, by [FHH, Exercise V.5.22], $N := q^{-1}(c_0)$ is a norming subspace of $X^*$ for $(X, \|\cdot\|)$, in particular $w^*$-dense in $X^*$. We can define on $X$ an equivalent norm $|\cdot|$ in such a way that $N$ is 1-norming for $(X, |\cdot|)$; precisely, $B(X, |\cdot|) := \overline{B(X, \|\cdot\|)}^{\sigma(X,N)}$. The topology $\sigma(X, N)$ on $X$ of the pointwise convergence on $N$ obviously induces on $\ell_1$ the topology $\sigma(\ell_1, c_0)$. Let $|||\cdot|||$ be an equivalent norm on $\ell_1$ such that the set $A(\ell_1, |||\cdot|||)$ has an empty interior. Such a norm exists by Theorem 23, and it is a dual norm. We may and do assume that $B(\ell_1, |||\cdot|||) \supset B(X, |\cdot|) \cap \ell_1$. Observe that $B(\ell_1, |||\cdot|||)$ is $\sigma(\ell_1, c_0)$-compact (and so $\sigma(X, N)$-compact), and that $B(X, |\cdot|)$ is $\sigma(X, N)$-closed. It is trivial then that

$$W := B(\ell_1, |||\ |||) + B(X, |\cdot|)$$

is a bounded absolutely convex and $\sigma(X,N)$-closed subset of $X$ containing the closed unit ball $B(X,|\cdot|)$, and so it is the closed unit ball of an equivalent norm $p$ on $X$. Now, let $x^* \in A(X,p)$. It is clear that its restriction $q(x^*)$ to $\ell_1$ belongs to $A(\ell_1, |||\cdot|||)$. Assume for a moment that $A(X,p)$ had a non-empty interior. The restriction mapping $q:(X^*,p) \to (\ell_\infty, |||\cdot|||)$ is continuous and onto, so an open mapping, taking open sets onto open sets. We should have then that $A(\ell_1, |||\cdot|||)$ has a non-empty interior, a contradiction. □

The previous result, Theorem 21 and Rosenthal's $\ell_1$ Theorem can be used as the main ingredients to deduce validity of Namioka's conjecture mentioned in 4.2.1 in the case of separable Banach spaces.

**Theorem 25.** ([AM, Thm. 3]) *If a separable Banach space $X$ is not reflexive then it has an equivalent norm $|||\cdot|||$ such that $A(X, |||\cdot|||)$ has an empty norm-interior.*

The spaces which can be renormed such that the set of norm attaining functionals has an empty norm-interior have the following stability property:

**Proposition 26.** *Assume that $(X, \|\cdot\|)$ is a Banach space and $Y \subset X$ is a complemented subspace of $(X, \|\cdot\|)$ such that $Y$ admits an equivalent norm $|||\cdot|||$ satisfying that $A(Y, |||\cdot|||)$ has an empty norm-interior. Then $|||\cdot|||$ can be extended to a norm $|||\cdot|||$ in $X$ with the same property.*

**Proof.** Let $X = Y \bigoplus M$ be a topological direct sum. Define a norm on $X$ by

$$|||y + m||| := \max\{|||y|||, \|m\|\} \qquad (y \in Y, m \in M).$$

This norm induces $|||\cdot|||$ on $Y$. Of course, $X^* = Y^* \bigoplus M^*$ and the dual norm is given by

$$|||y^* + m^*||| = |||y^*||| + \|m^*\| \qquad (y^* \in Y^*,\ m^* \in M^*),$$

where we used the same symbol to denote a norm and the corresponding dual norm. A functional $x^* = y^* + m^*$ attains its norm if and only if both $y^*$ and $m^*$ attain their corresponding norm, that is

$$A(X, |||\cdot|||) = A(Y, |||\cdot|||) + A(M, |||\cdot|||).$$

Since any ball in $X^*$ contains a product of balls of $Y^*$ and $M^*$ and, by assumption, the set $A(Y, |||\cdot|||)$ has an empty norm-interior, then the subset $A(X, |||\cdot|||)$ has also an empty norm-interior. □

Recall that a Banach space $X$ is *weakly Lindelöf determined* (in short, WLD), if $(B_{X^*}, w^*)$ is a *Corson compact*, i.e., a compact subspace of a product of lines such that every element has only a countable number of non-zero coordinates. Every *weakly compactly generated* Banach space (i.e., a Banach space with a weakly compact linearly dense subset) is WLD. It is well known that a WLD Banach space $X$ has the *separable complementation property*, i.e., the fact that for every separable subspace $Y \subset X$ there exists a separable space $Z$ such that $Y \subset Z \subset X$ and $Z$ is complemented in $X$ (for these concepts and properties see, for example, [Fa, Chap. 7] and [FHH, Chap. 12]).

The following result extends the class of spaces where Namioka's conjecture holds true:

**Theorem 27.** *If a non-reflexive Banach space $X$ has the separable complementation property, in particular, if $X$ is WLD, then it has an equivalent norm $|||\cdot|||$ such that $A(X, |||\cdot|||)$ has an empty norm-interior.*

**Proof.** If $X$ is not reflexive, it contains, by the Eberlein-Šmulyan Theorem, a non-reflexive and separable closed subspace $Y$. There exists then a complemented and separable subspace $Z$ containing $Y$, so in particular $Z$ is not reflexive either. It follows from Theorem 25 that $Z$ can be renormed with a norm $|||\cdot|||$ such that $A(Z, |||\cdot|||)$ has an empty norm-interior. We apply now Proposition 26 to define a norm $|||\cdot|||$ on $X$ with the same property. □

**Remark.** There are Banach spaces $X$ with the separable complementation property and without a Projectional Resolution of the Identity; in particular, they are not WLD, see [DGZ, Definition VI.1.1 and Example VI.8.6].

## 5. Open Problems

There were several questions disseminated along this paper. We collect all of them below.

1. Is Namioka's conjecture true in the case of a general (non-separable) Banach space?
2. Let $(X, \|\cdot\|)$ be a Banach space such that every $x^* \in X^*$ attains the supremum on $B_{(X,\|\cdot\|)}$. Prove, without using James Theorem, that for every equivalent norm on $X$ the same is true.
3. Let $(X, \|\cdot\|)$ be a Banach space with a non-dentable unit ball. Is the set $A(X, \|\cdot\|)$ $1^{\text{st}}$ category?
4. Let $(X, \|\cdot\|)$ be a Banach space. Assume that for every nonempty norm-closed, bounded and convex subset $A$ of $X^*$ there exists $x \in X$ which attains its supremum on $A$. Is $X$ Asplund?

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