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## Some Remarks on Non-Special Coherent Aronszajn Trees

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We introduce some guessing principles sufficient for the existence of non-special coherent Aronszajn trees and show how they relate to some of the standard principles in Set Theory (like  $MA_{\omega_1}$  and  $\hat{\diamond}$ ).

A variant of a question of I. Juhasz asks whether the principle  $\clubsuit$  implies the existence of a non-special Aronszajn tree. Motivated by this question, we investigate when a coherent Aronszajn tree associated with the  $\varrho_1$  function of Todorčević (see [5]) is not special. To do this, we define principles  $\star_0$  and  $\star_1$  and their corresponding weak versions  $w\star_0$  and  $w\star_1$ . The principles  $\star_0$  and  $\star_1$  are strong enough to construct non-special coherent Aronszajn trees. All these principles are weak in the sense that they are all consistent with  $MA_{\sigma\text{-centered}}$  and some of them are strong in the sense that they do not follow from  $\hat{\diamond}$ .

Our notation is mostly standard (see Kunen [4] and Jech [2]). We will use  $\Lambda$  to denote the collection of all countable limit ordinals.  $A \sqsubseteq B$  will be used to denote that  $A$  is an initial segment of  $B$ , whenever  $A, B$  are subsets of  $\omega_1$ . If  $A$  is a subset of  $\omega_1$ , we will use  $ot(A)$  to denote the order-type of  $A$ . The symbol  $\frown$  denotes concatenation.

By a  $C$ -sequence (see [5]) we mean a sequence  $\langle C_\alpha : \alpha \in \omega_1 \rangle$  with the following properties:  $C_{\alpha+1} = \{\alpha\}$ ,  $C_\alpha$  is a cofinal subset of  $\alpha$  of order-type  $\omega$ , whenever  $\alpha$  is a countable limit ordinal  $> 0$ .

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**Definition 1.** The principles  $\star_1$ ,  $w\star_1$ ,  $\star_0$   $w\star_0$  are defined as follows:

- $\star_0$  There is a  $C$ -sequence  $\langle S_\alpha : \alpha \in \omega_1 \rangle$  such that for every  $\varphi : \Lambda \rightarrow \omega$  there are  $\alpha, \beta \in \Lambda$  such that  $\varphi(\alpha) = \varphi(\beta)$ ,  $S_\beta \cap \alpha \sqsubseteq S_\alpha$  and  $\alpha \in S_\beta$ .
- $w\star_0$  There is a  $C$ -sequence  $\langle S_\alpha : \alpha \in \omega_1 \rangle$  such that for every  $\varphi : \Lambda \rightarrow \omega$  there are  $\alpha, \beta \in \Lambda$  such that  $\varphi(\alpha) = \varphi(\beta)$  and  $\alpha \in S_\beta$ .
- $\star_1$  There is a  $C$ -sequence  $\langle S_\alpha : \alpha \in \omega_1 \rangle$  such that for every stationary set  $S$  there are  $\alpha, \beta \in S$  such that  $S_\beta \cap \alpha \sqsubseteq S_\alpha$  and  $\alpha \in S_\beta$ .
- $w\star_1$  There is a  $C$ -sequence  $\langle S_\alpha : \alpha \in \omega_1 \rangle$  such that for every stationary set  $S$  there are  $\alpha, \beta \in S$  such that  $\alpha \in S_\beta$ .

Following [5], to every  $C$ -sequence  $\langle C_\alpha : \alpha < \omega_1 \rangle$  we associate two functions  $\varrho_0, \varrho_1$ . The function  $\varrho_0 = \varrho_0(C_\alpha : \alpha < \omega_1) : [\omega_1]^2 \rightarrow \omega^{<\omega}$  is defined recursively as follows

$$\varrho_0(\alpha, \beta) = \begin{cases} \langle |C_\beta \cap \alpha| \rangle \frown \varrho_0(\alpha, \min(C_\beta \setminus \alpha)) & \text{if } \alpha < \beta \\ \emptyset & \text{if } \alpha = \beta \end{cases}.$$

Even though,  $\varrho_0$  is an important function on its own, we use it only as an auxiliary tool in some proofs of the theorems in this article.

The function  $\varrho_1 = \varrho_1(C_\alpha : \alpha < \omega_1) : [\omega_1]^2 \rightarrow \omega$  is defined recursively by

$$\varrho_1(\alpha, \beta) = \begin{cases} \max \{ |C_\beta \cap \alpha|, \varrho_1(\alpha, \min(C_\beta \setminus \alpha)) \} & \text{if } \alpha < \beta \\ 0 & \text{if } \alpha = \beta \end{cases},$$

Thus,  $\varrho_1(\alpha, \beta)$  is simply the maximal integer appearing in the sequence  $\varrho_0(\alpha, \beta)$ . We will focus on the function  $\varrho_1$ . Basic properties of the  $\varrho_1$  function are mentioned in the next lemma.

**Lemma 2** (Todorćević [5]). For all  $\alpha < \beta < \omega_1$  and  $n < \omega$ ,

- (a)  $\{ \xi \leq \alpha : \varrho_1(\xi, \alpha) \leq n \}$  is finite,
- (b)  $\{ \xi \leq \alpha : \varrho_1(\xi, \alpha) \neq \varrho_1(\xi, \beta) \}$  is finite.

Let  $\varrho_{1\alpha} : \alpha \rightarrow \omega$  be defined by  $\varrho_{1\alpha}(\xi) = \varrho_1(\xi, \alpha)$  for every  $\xi < \alpha$ . Then it follows from the previous lemma that the sequence

$$\varrho_{1\alpha} : \alpha \rightarrow \omega \ (\alpha < \omega_1)$$

of finite-to-one functions is *coherent* in the sense that  $\varrho_{1\alpha} = {}^*\varrho_{1\beta} \upharpoonright \alpha$  whenever  $\alpha \leq \beta$ . (Here  $=^*$  means the fact that the functions agree on all but finitely many arguments). The corresponding tree

$$T(\varrho_1) = \{ \varrho_{1\beta} \upharpoonright \alpha : \alpha < \beta \leq \omega \}$$

is a coherent Aronszajn tree.

The following two theorems show the relevance of the guessing principles  $\star_0$  and  $\star_1$ .

**Theorem 3.**  $\star_0$  implies that there is a non special coherent Aronszajn tree.

*Proof.* Let  $T = T(\varrho_1)$  be the coherent Aronszajn tree constructed from a  $\star_0$ -sequence  $\langle S_x : \alpha < \omega_1 \rangle$  i.e.  $\varrho_1 = \varrho_1(S_x : \alpha < \omega_1)$ . To prove the theorem it is enough to check that  $A = \{\varrho_1 : \alpha \in \Lambda\} \subseteq T$  is not a countable union of antichains. Given any partition  $\varphi : A \rightarrow \omega$  of  $A$ , we define a new function  $\hat{\varphi} : \Lambda \rightarrow \omega$  by  $\hat{\varphi}(\alpha) = \varphi(\varrho_{1\alpha})$  for every  $\alpha \in \Lambda$ . It follows, using  $\star_0$ , that there are  $\alpha, \beta \in \Lambda$  such that  $\hat{\varphi}(\alpha) = \hat{\varphi}(\beta)$ ,  $S_\beta \cap \alpha \sqsubseteq S_\alpha$  and  $\alpha \in S_\beta$ . Then let us check that  $\varrho_{1\alpha} \subseteq \varrho_{1\beta}$ . Let  $\{\xi_k : k \leq n\}$  be the increasing enumeration of  $S_\beta \cap \alpha$ . The proof proceeds by cases:

**Case 1.** If  $\xi \in [0, \xi_0]$  then  $\varrho_0(\xi, \beta) = \langle 0 \rangle \frown \varrho_0(\xi, \xi_0)$ . Since  $S_\beta \cap \alpha \sqsubseteq S_\alpha$  the same holds for  $\varrho_0(\xi, \alpha)$ . Then by the definition of  $\varrho_1$  we have that  $\varrho_1(\xi, \alpha) = \varrho_1(\xi, \beta)$ .

**Case 2.** If  $\xi \in (\xi_k, \xi_{k+1}]$  then  $\varrho_0(\xi, \beta) = \langle |S_\beta \cap \xi| \rangle \frown \varrho_0(\xi, \min(S_\beta \setminus \xi))$ . However,  $S_\beta \cap \alpha \sqsubseteq S_\alpha$  implies that  $\xi_{k+1} = \min(S_\beta \setminus \xi) = \min(S_\alpha \setminus \xi)$  and  $|S_\beta \cap \xi| = |S_\alpha \cap \xi|$  so  $\varrho_1(\xi, \beta) = \varrho_1(\xi, \alpha)$ .

**Case 3.**

If  $\xi \in (\xi_n, \alpha)$  then  $\varrho_0(\xi, \beta) = \langle n \rangle \frown \varrho_0(\xi, \alpha)$ , and  $\varrho_0(\xi, \alpha) = \langle |S_\alpha \cap \xi| \rangle \frown \varrho_0(\xi, \min(S_\alpha \setminus \xi))$ . However, since  $S_\beta \cap \alpha \sqsubseteq S_\alpha$ ,  $n \leq |S_\alpha \cap \xi|$  so we have that  $\varrho_1(\xi, \alpha) = \varrho_1(\xi, \beta)$ .

Then  $\forall \xi < \alpha (\varrho_{1\alpha}(\xi) = \varrho_{1\beta}(\xi))$ . So we are done.  $\square$

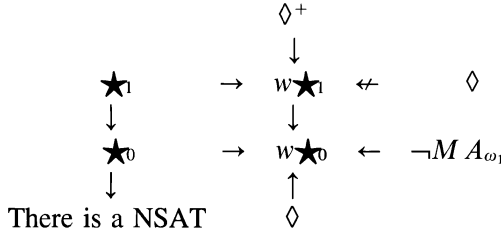
**Theorem 4.**  $\star_1$  implies that there is a coherent Aronszajn tree  $T$  which does not have stationary antichains.

*Proof.* Let  $T = T(\varrho_1)$  be the coherent Aronszajn tree constructed from a  $\star_1$ -sequence  $\langle S_x : \alpha < \omega_1 \rangle$  i.e.  $\varrho_1 = \varrho_1(S_x : \alpha < \omega_1)$ . The result follows using the same argument as in the previous theorem and the following claim.

**Claim.**  $T$  has a stationary antichain if and only if  $\{\varrho_{1\alpha} : \alpha \in \omega_1\}$  has one.

Let us prove the claim. let  $A = \{t_\alpha : \alpha \in S\}$  be a stationary antichain of  $T$ , we may assume that  $|T_\alpha \cap A| = 1$  and  $ht(t) = \alpha$  for every  $\alpha \in S$ . Note that  $S$  is a stationary set. For each  $t_\alpha \in A$  there is an  $F_\alpha \in [\alpha]^{<\omega}$  such that  $t_\alpha(\xi) = \varrho_{1\alpha}(\xi)$  for every  $\xi \in (\alpha \setminus F_\alpha)$ . By the pressing down lemma, we can find a stationary set  $S' \subseteq S$  such that  $F_\alpha = F$  for every  $\alpha \in S'$ . Using again the pressing down lemma we can find a stationary set  $\hat{S} \subseteq S'$  such that  $t_\alpha \upharpoonright F = t_\beta \upharpoonright F$  for every  $\alpha < \beta \in \hat{S}$ . Then  $\forall \alpha < \beta \in \hat{S}$  there is a  $\xi \in (\alpha \setminus F)$  such that  $t_\alpha(\xi) \neq t_\beta(\xi)$ . This implies that  $t_\alpha(\xi) = \varrho_{1\alpha}(\xi) \neq \varrho_{1\beta}(\xi) = t_\beta(\xi)$ . So  $\{\varrho_{1\alpha} : \alpha \in S\}$  is a stationary antichain in  $\{\varrho_{1\alpha} : \alpha < \omega_1\}$ , and this finishes the proof.  $\square$

As we have seen, the principles  $\star_0$  and  $\star_1$  are guessing principles which imply the existence of non-special Aronszajn trees. In order to have a better understanding of these principles we will compare them with some well known principles in set theory, summed up in the following diagram.



Here *NSAT* is an abbreviation for *non-special Aronszajn tree*. As the following theorem shows all the principles are relatively consistent with *ZFC*, even with  $MA_{\sigma\text{-centered}}$ .

**Theorem 5.** *If  $V[G]$  is the generic extension obtained by adding a single Cohen real then  $V[G] \models \star_1$ .*

*Proof.* From now on assume that  $c : \omega \rightarrow [\omega]^{<\omega}$  is a Cohen-generic real and  $e_\alpha : \alpha \rightarrow \omega$  ( $\alpha < \omega_1$ ) is a coherent sequence of finite-to-one functions. Let  $\langle C_\alpha : \alpha < \omega_1 \rangle$  be an arbitrary *C*-sequence. We change this *C*-sequence to a *C*-sequence  $\langle S_\alpha : \alpha < \omega_1 \rangle$  in the following way:

$$S_\alpha = \{ \xi < \alpha : C_\alpha(n) \leq \xi < C_\alpha(n+1), e_\alpha(\xi) \in c(n) \},$$

where  $C_\alpha(0) = 0$  and  $C_\alpha(n)$  is the  $n$ th element of  $C_\alpha$  for  $0 < n < \omega$ . Note that since  $e_\alpha$ 's are finite-to-one  $ot(S_\alpha) = \omega$ . Let us check that  $\langle S_\alpha : \alpha < \omega_1 \rangle$  is a  $\star_1$ -sequence. Assume that  $A$  is a stationary subset of  $\omega_1$ . Note that if  $A$  is stationary in  $V[G]$ , then there is a stationary set  $A_0 \in V$  such that  $A_0 \subset A$ . So without loss of generality we may assume that  $A$  is in the ground model. Fix  $p \in Fn(\omega, [\omega]^{<\omega})$  with  $dom(p) \in \omega$ , use the pressing down lemma to find a stationary set  $S \subset A$  such that  $S_\alpha$  agree with  $S_\beta$  in all the places decided by  $p$  for every  $\alpha, \beta \in S$ . Pick an accumulation point  $\beta$  of  $S$ , now choose an  $\alpha \in S$  in such a way that  $C_\beta(n_0) < \alpha \leq C_\beta(n_0 + 1)$  where  $dom(p) < n_0$ . Let  $q$  be defined by

$$q(n) = \begin{cases} p(n) & \text{if } n \in dom(p) \\ \emptyset & \text{if } dom(p) < n < n_0 \\ \{e_\beta(\alpha)\} & \text{if } n = n_0 \end{cases}$$

then  $q \Vdash \text{``}\dot{S}_\beta \cap \alpha \sqsubseteq \dot{S}_\alpha \ \& \ \alpha \in \dot{S}_\beta\text{''}$ . □

**Corollary 6.**  $\star_1$  (and hence also  $\star_0$ ,  $w\star_0$  and  $w\star_1$ ) are relatively consistent with  $MA_{\sigma\text{-centered}}$ .

*Proof.* Let  $V$  be a model of  $MA$  and  $\mathbb{P}$  a forcing which adds a single Cohen real. By the previous theorem if  $G$  is a  $\mathbb{P}$ -generic filter then  $M[G] \models \star_1$  and by the theorem of Roitman (see [1]) the extension  $M[G] \models MA_{\sigma\text{-centered}}$ . □

The fact that after adding a single Cohen real there is a coherent Aronszajn tree without stationary antichains was first observed by B. König in [3]. The following

propositions give us some relationship between  $\diamond$  and  $\diamond^+$  with our guessing principles.

**Proposition 7.**  $\diamond$  implies  $w\star_0$ .

*Proof.* Let  $\langle \varphi_\alpha : \alpha \in \omega_1 \rangle$  be a  $\diamond$ -sequence which guesses elements of  $\omega^{\omega_1}$  (i.e.  $\varphi_\alpha \in \omega^\alpha$ ). Define  $X_\alpha = \{n : \varphi_\alpha^{-1}(n) \text{ is cofinal in } \alpha\}$  for every limit  $\alpha$ . For every  $\alpha \in \Lambda$  choose  $S_\alpha \subseteq \alpha$  of order type  $\omega$  such that  $S_\alpha \cap \varphi_\alpha^{-1}(n)$  is a cofinal in  $\alpha$  for every  $n \in X_\alpha$ . This is very easy to do. Let us check that the  $C$ -sequence  $\langle S_\alpha : \alpha < \omega_1 \rangle$  has the required properties. Now, let  $\varphi : \Lambda \rightarrow \omega$  be given. Set  $X = \{n \in \omega : \varphi^{-1}(n) \text{ is cofinal in } \omega_1\}$  and  $C = \{\alpha : \forall n \in X (\varphi^{-1}(n) \text{ is cofinal in } \alpha)\}$ . It is easy to see that  $C$  is a club in  $\omega_1$ . Let  $\xi_0 = \max\{\varphi^{-1}(n) : n \notin X\} + 1$  and  $S = \{\alpha : \varphi_\alpha = \varphi \upharpoonright \alpha\}$ . Pick any  $\beta \in C \cap S \cap [\xi_0, \omega_1)$  then  $\varphi(\beta) = n_0 \in X_\beta$ . It follows from the properties of  $S_\beta$  that there is an  $\alpha \in S_\beta$  such that  $\varphi(\alpha) = n_0$ .  $\square$

**Proposition 8.**  $\diamond^+$  implies  $w\star_1$ .

*Proof.* Let  $\langle \mathcal{A}_\alpha : \alpha \in \omega_1 \rangle$  be a  $\diamond^+$ -sequence. For each  $\alpha$ , let  $S_\alpha \subset \alpha$  be a sequence of order-type  $\omega$  such that  $S_\alpha \cap A \neq \emptyset$  for every  $A \in \mathcal{A}_\alpha$  (this can be done by an easy induction). Let us verify that  $\langle S_\alpha : \alpha \in \omega_1 \rangle$  is a  $w\star_1$ -sequence. Given a stationary set  $S$ , there is a club  $C$  such that  $\forall \alpha \in C (S \cap \alpha \in \mathcal{A}_\alpha)$ . Pick any  $\beta \in (C \cap S)$  then  $S_\beta \cap (S \cap \beta) \neq \emptyset$ , now choose  $\alpha \in S_\beta \cap (S \cap \beta)$ . Then  $\alpha, \beta \in S_\beta$ . So we are done.  $\square$

We do not know if in the previous propositions we can replace the weak versions for the stronger ones. However, we have some limitations as the following theorem shows.

**Theorem 9.**  $\diamond$  does not imply  $w\star_1$ .

To prove the theorem we need the following lemmas.

**Lemma 10.** For every  $C$ -sequence  $\langle S_\alpha : \alpha \in \omega_1 \rangle$  there is an  $\alpha$  such that for every  $\beta > \alpha$ ,  $\{\gamma : (S_\gamma \setminus \alpha) \cap \beta = \emptyset\}$  is stationary.

*Proof.* Suppose that this is not the case. Then for every  $\alpha$  there is a  $\beta(\alpha)$  and a club  $C_\alpha$  such that  $(S_\gamma \setminus \alpha) \cap \beta(\alpha) \neq \emptyset$ , whenever  $\gamma \in C_\alpha$ . Pick  $\alpha_0 \in \omega_1$  and define  $\alpha_{n+1} = \beta(\alpha_n)$ . Let  $\xi \in \bigcap_{n \in \omega} C_{\alpha_n}$  be greater than  $\alpha = \sup\{\alpha_n : n \in \omega\}$ . Since  $S_\xi$  intersects each interval  $[\alpha_n, \alpha_{n+1})$ ,  $\alpha$  is an accumulation point of  $S_\xi$ , so the order-type of  $S_\xi$  is greater than  $\omega$ , which is a contradiction.  $\square$

The following lemma is a well known fact.

**Lemma 11.** (1) Countable support iteration of  $\sigma$ -closed forcings is  $\sigma$ -closed,  
(2) Every  $\sigma$ -closed forcing preserves  $\diamond$ .

*Proof of theorem 9.* For every  $C$ -sequence  $\mathcal{C} = \langle C_\alpha : \alpha \in \omega_1 \rangle$ , define the notion of forcing  $\mathbb{P}_\mathcal{C}$  where

$$\mathbb{P}_\mathcal{C} = \{p \in 2^{<\omega} : \forall \alpha \in p^{-1}(1), C_\alpha \cap p^{-1}(1) = \emptyset \text{ and } p \upharpoonright \alpha_\mathcal{C} \equiv 0\}$$

Here  $\alpha_{\mathcal{C}}$  is the  $\alpha$  in the previous lemma which correspond to the  $C$ -sequence  $\mathcal{C}$ , and the order is by extension.

**Claim 1.**  $\mathbb{P}_{\mathcal{C}}$  is a  $\sigma$ -closed forcing.

Let  $p_n$  be a decreasing sequence of conditions in  $\mathbb{P}_{\mathcal{C}}$  and set  $p_{\omega} = \bigcup_{n \in \omega} p_n$ . Obviously,  $p_{\omega} \in 2^{<\omega_1}$  and  $p_{\omega} \upharpoonright \alpha_{\mathcal{C}} \equiv 0$ . Suppose that there are  $\alpha, \beta \in \pi_{\omega}^{-1}(1)$  such that  $\alpha \in C_{\beta}$  i.e.  $C_{\beta} \cap p_{\omega}^{-1}(1) \neq \emptyset$ , then there are  $n, m \in \omega$  such that  $\alpha \in \text{dom}(p_n)$  and  $\beta \in \text{dom}(p_m)$  but this implies that  $\alpha, \beta \in p_{m+n}^{-1}(1)$  and  $C_{\beta} \cap p_{m+n}^{-1}(1) \neq \emptyset$  which is a contradiction.

**Claim 2.**  $\mathbb{P}_{\mathcal{C}}$  forces that  $\mathcal{C}$  is not a  $w\star_1$ -sequence.

Let  $f_G$  be the  $\mathbb{P}_{\mathcal{C}}$ -generic function and  $S = f_G^{-1}(1)$ . To see that  $\mathcal{C}$  is not a witness for  $w\star_1$  in  $M[G]$  it suffices to prove that  $S$  is stationary in  $M[G]$ . Let  $\dot{C}$  be a name for a club and  $p \in \mathbb{P}_{\mathcal{C}}$  a condition such that  $p \Vdash \text{“}\dot{C} \text{ is a club”}$ . By Lemma 10, we can find a sequence  $M_0 \subseteq M_1 \subseteq \dots \subseteq M_n \subseteq \dots$  of countable elementary submodels of  $H(\xi)$  for  $\xi$  large enough, such that  $p, \langle C_{\alpha} : \alpha \in \omega_1 \rangle, \dot{C} \in M_0$  and moreover,  $(C_{\delta_n} \setminus a_{\mathcal{C}}) \cap \text{dom}(p) = \emptyset$ , where  $\delta_n = M_n \cap \omega_1$  and we may assume that  $\delta_n \in M_{n+1}$ . Set  $M_{\omega} = \bigcup_{n \in \omega} M_n$  and  $\delta = M_{\omega} \cap \omega_1$ . We will construct a sequence  $p_n$  of conditions such that  $p_{n+1} \leq p_n$ ,  $p_n \Vdash \text{“}\delta_n \in \dot{C}\text{”}$ ,  $p_n^{-1}(1) \cap C_{\delta} = \emptyset$  and  $p_n \in M_n$  by recursion as follows:

Let  $\xi_0 = \max(C_{\delta} \cap \delta_0)$ , and extend  $p$  to a condition  $q = p \cup \{(\alpha, 0) : \alpha \in [\text{dom}(p), \xi_0]\}$ . Note that  $q \in M_0$ . Since  $M_0[G] \vDash \text{“}\dot{C} \text{ is a club”}$  there is an  $\eta_0 \in \omega_1 \cap M_0$  and a  $p_0 \in \mathbb{P}_{\mathcal{C}} \cap M_0$ ,  $p_0 \leq q$  such that  $p_0 \Vdash \text{“}\eta_0 \in \dot{C}\text{”}$ .

For the inductive step assume that we have constructed  $p_k$  for  $k \leq n$  with the required properties. Pick  $\xi_{n+1} < \delta_{n+1}$  such that  $\xi_{n+1} > \max(C_{\delta} \cap \delta_{n+1})$ . Then  $q = p_n \cup \{(\alpha, 0) : \alpha \in [\text{dom}(p_n), \xi_{n+1}]\} \in M_{n+1}$  is a condition. As  $q \Vdash \text{“}\dot{C} \text{ is a club”}$  there is a  $\eta_{n+1} < \delta_{n+1}$  and a condition  $q \geq p_{n+1} \in M_{n+1}$  such that  $p_{n+1} \Vdash \text{“}\eta_{n+1} \in \dot{C}\text{”}$ .

Finally, let

$$p_{\omega} = \bigcup_{n \in \omega} p_n \cup \{(\delta, 1)\}.$$

Note that  $p_{\omega}$  is a condition as  $p_{\omega}^{-1}(1) \cap C_{\delta} = \emptyset$ . As  $p_{\omega} \leq p_n$  for all  $n \in \omega$ ,  $p_{\omega} \Vdash \text{“}\{\eta_n : n \in \omega\} \subseteq \dot{C}\text{”}$ . As  $\delta = \sup_{n \in \omega} \eta_n$  and since  $\dot{C}$  is a name for a club  $p_{\omega} \Vdash \text{“}\delta \in \dot{C} \cap \dot{C}\text{”}$ . So  $S$  is stationary and Claim 2 holds.

Let  $V = L$  and construct a countable support iteration  $\mathbb{P} = \langle \mathbb{P}_{\alpha}, \dot{Q}_{\alpha} : \alpha < \omega_2 \rangle$  so that  $\Vdash_{\mathbb{P}_{\alpha}} \text{“}\dot{Q}_{\alpha} = \mathbb{P}_{\mathcal{C}} \text{ for some } C\text{-sequence } \mathcal{C}\text{”}$ . By a standard book-keeping argument one can make sure that all  $C$ -sequences in the intermediate models are listed. Let  $G$  be a  $\mathbb{P}$ -generic. Since every  $C$ -sequence  $\mathcal{C}$  in  $M[G]$  has a  $\mathbb{P}_{\alpha}$ -name for some  $\alpha < \omega_2$ , and at some stage  $\beta < \omega_2$  we have that  $\dot{Q}_{\beta} = \mathbb{P}_{\mathcal{C}}$  then  $\mathcal{C}$  is not a  $w\star_1$ -sequence. So  $M[G] \vDash \neg w\star_1$  and by the Lemma 11.  $M[G] \vDash \dot{\diamond}$ .

Finally we show that none of the principles is consistent with Martin's Axiom.

**Theorem 12.**  $MA(\omega_1)$  implies  $\neg w\star_0$ .

*Proof.* Let  $\langle C_\alpha : \alpha < \omega_1 \rangle$  be a  $C$ -sequence. Define

$$\mathbb{P} = \{p : A \rightarrow \omega : A \in [\Lambda]^{<\omega}, (\forall \alpha < \beta)(p(\alpha) = p(\beta) \rightarrow \alpha \notin C_\beta)\}$$

ordered by inverse inclusion. It is easy to see that, if  $f_G$  is the generic function, then  $f_G$  is defined on  $\Lambda$  and forces that  $\langle C_\alpha : \alpha < \omega_1 \rangle$  is not a  $w\star_0$ -sequence, to assure both we need to meet only  $\omega_1$  many dense sets. To finish the proof it suffices to check that:

**Claim**  $\mathbb{P}$  is a c.c.c. forcing.

Suppose that  $\{p_\alpha : \alpha \in \omega_1\}$  is an antichain. By a standard  $\Delta$ -system type argument, we can assume that their domains form a  $\Delta$ -system with root  $r$ , such that there is a  $N \in \omega$  with  $|dom(p_\alpha)| = N$  for each  $\alpha \in \omega_1$  and all the functions agree on  $r$ . Moreover, we can assume that  $dom(p_\alpha) \cap dom(p_\beta) = \emptyset$  for every  $\alpha, \beta \in \omega_1$ , and  $\max(dom(p_\alpha)) < \min(dom(p_\beta))$  if  $\alpha < \beta$ . Now, set  $dom(p_{\omega \cdot N + 1}) = \{\xi_1, \dots, \xi_N\}$ . Since  $p_{\omega \cdot N + 1}$  is incompatible with  $p_\alpha$  for every  $\alpha < \omega \cdot N + 1$ ,  $(\bigcup_{i=1}^N C_{\xi_i}) \cap dom(p) \neq \emptyset$  for every  $\alpha < \omega \cdot N + 1$ . Then by the pigeon hole principle there is a  $i$  such that  $ot(C_{\xi_i}) \geq \omega + 1$ . However, this contradicts the fact that  $\langle C_\alpha : \alpha < \omega_1 \rangle$  is a  $C$ -sequence, so we are done.  $\square$

We conclude with some open problems.

**Question 13.** (1) Does  $w\star_1$  imply  $\star_1$ ?

(2) Does  $w\star_0$  imply  $\star_0$ ?

(2) Does  $\clubsuit$  imply  $\star_0$ ?

An earlier version of this paper contained also the following questions: (4) Does  $\diamond^+$  imply  $\star_1$ ? and (5) Does  $\diamond$  imply  $\star_0$ ? These questions were answered by Paul Larson. We present the proofs with his kind permission.

**Theorem 14** (Larson). (i)  $\diamond$  implies  $\star_1$ .

(ii)  $\diamond^+$  implies  $\star_1$ .

*Proof.* Fix for every limit ordinal  $\alpha < \omega_1$  a strictly increasing sequence  $\{\alpha_n : n \in \omega\}$  such that  $\sup_{n \in \omega} \alpha_n = \alpha$  and let  $\theta$  be a sufficiently large regular cardinal.

To prove (i) let  $\langle \varphi_\alpha : \alpha < \omega_1 \rangle$  be a  $\diamond$ -sequence which guesses elements of  $\omega^{\omega_1}$  (i.e.  $\varphi_\alpha \in \omega^\alpha$ ). Construct recursively a  $C$ -sequence  $\langle S_\alpha : \alpha \in \Lambda \rangle$  and a sequence  $\langle e_n^\alpha : \alpha \in \Lambda, n \in \omega \rangle$  of finite subsets of  $\alpha$  with the following properties:

(i)  $S_\alpha = \bigcup_{n \in \omega} e_n^\alpha$ ,

(ii)  $e_n^\alpha \sqsubseteq e_{n+1}^\alpha$ ,  $\max(e_{n+1}^\alpha) > \alpha_n$ ,

(iii)  $e_{n+1}^\alpha = e_n^\alpha \cup \{\xi\}$ , where

$\xi = \min\{\eta : \eta > \max(e_n^\alpha \cup \{\alpha_n\}) \wedge e_n^\alpha \sqsubseteq S_\eta \wedge \varphi_\alpha(\xi) = n\}$  if such  $\xi$  exists, otherwise  $\xi = \alpha_N$ , where  $N = \min\{k : \alpha_k > (e_n^\alpha \cup \{\alpha_n\})\}$ .

Now let us check that  $\langle S_\alpha : \alpha \in \Lambda \rangle$  is a  $\star_0$ -sequence. It follows from (ii) that  $S_\alpha$  is cofinal in  $\alpha$  with order type  $\omega$  for every limit  $\alpha < \omega_1$ . Let  $\varphi : \omega_1 \rightarrow \omega$  be given. Set  $S = \{\alpha < \Lambda : \varphi_\alpha = \varphi \upharpoonright \alpha\}$ , since  $\langle \varphi_\alpha : \alpha < \omega_1 \rangle$  is a  $\diamond$ -sequence  $S$  is a stationary set. Since  $C = \{\omega_1 \cap M : M \prec H(\theta)\}$  such that  $\varphi, \langle S_\alpha : \alpha < \Lambda \rangle$ ,



$\langle \varphi_\alpha : \alpha < \omega_1 \rangle \in M$  is a club there is an  $M \in C$  such that  $M \cap \omega_1 = \delta \in S$ . Suppose that  $\varphi(\delta) = n$  and let  $e_{n-1}^\delta = S_\delta \upharpoonright n - 1$  (here  $e_{-1}^\delta = \emptyset$ ) then for every  $\alpha \in M$

$$H(\theta) \vDash \exists \beta > \alpha (\varphi(\beta) = n \wedge e_{n-1}^\delta \sqsubseteq S_\beta).$$

So, there is an  $\alpha \in \delta$ ,  $\alpha > (e_n^\delta \cup \{\alpha_n\})$  such that  $n = \varphi(\alpha) = \varphi_\delta(\alpha)$  and  $e_{n-1}^\delta \sqsubseteq S_\alpha$ . It follows for the construction of  $e_{n+1}^\delta$  that  $e_{n+1}^\delta = e_n^\delta \cup \{\xi\}$  for some  $\xi$  with the same properties as  $\alpha$ . Then  $\varphi(\xi) = \varphi(\delta)$ ,  $\xi \in S_\delta$  and  $S_\delta \cap \alpha = e_{n+1}^\delta \sqsubseteq S_\xi$ .

To prove (ii) let  $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$  be a  $\diamond^+$ -sequence. Enumerate  $\mathcal{A}_\alpha$  as  $\{A_n^\alpha : n \in \omega\}$ . Construct recursively a  $C$ -sequence  $\langle S_x : \alpha \in \Lambda \rangle$  and a sequence  $\langle e_n^\alpha : \alpha \in \Lambda, n \in \omega \rangle$  of finite subsets of  $\alpha$  with the following properties:

$$(i) S_x = \bigcup_{n \in \omega} e_n^\alpha,$$

$$(ii) e_n^\alpha \sqsubseteq e_{n+1}^\alpha, \max(e_{n+1}^\alpha) > \alpha_n,$$

$$(iii) e_{n+1}^\alpha = e_n^\alpha \cup \{\xi\}, \text{ where}$$

$\xi = \min(\eta : \eta > \max(e_n^\alpha \cup \{a_n\}) \wedge e_n^\alpha \sqsubseteq S_\eta \wedge (\xi) \in A_{n+1}^\alpha)$  if such  $\xi$  exists, otherwise  $\xi = \alpha_N$ , where  $N = \min\{k : \alpha_k > (e_n^\alpha \cup \{\alpha_n\})\}$ .

Now let us check that  $\langle S_x : \alpha \in \Lambda \rangle$  is a  $\star_1$ -sequence. It follows from (ii) that  $S_x$  is cofinal in  $\alpha$  with order type  $\omega$ . Let  $S$  be stationary. Set  $D = \{\alpha < \Lambda : S \cap \alpha \in \mathcal{A}_\alpha\}$ , since  $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$  is a  $\diamond^+$ -sequence  $D$  is a club. Since  $C = \{\omega_1 \cap M : M \prec H(\theta) \text{ such that } S, \langle S_x : \alpha < \Lambda \rangle, \langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle \in M\}$  is a club there is an  $M \in C$  such that  $M \cap \omega_1 = \delta \in S \cap D$ . Suppose that  $S \cap \delta = A_n^\delta$  and let  $e_{n-1}^\delta = S_\delta \upharpoonright n - 1$  then for every  $\alpha \in M$

$$H(\theta) \vDash \exists \beta > \alpha (\beta \in S \wedge e_{n-1}^\delta \sqsubseteq S_\beta).$$

So, there is an  $\alpha \in \delta$ ,  $\alpha > (e_n^\delta \cup \{\alpha_n\})$  such that  $\alpha \in A_{n+1}^\delta$  and  $e_n^\delta \sqsubseteq S_\alpha$ . It follows for the construction of  $e_{n+1}^\delta$  that  $e_{n+1}^\delta = e_n^\delta \cup \{\xi\}$  for some  $\xi$  with the same properties as  $\alpha$ . Then we have that  $\xi \in S$ ,  $\xi \in S_\delta$  and  $S_\delta \cap \xi = e_{n-1}^\delta \sqsubseteq S_\xi$ .  $\square$

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