Acta Universitatis Carolinae. Mathematica et Physica

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*Acta Universitatis Carolinae. Mathematica et Physica*, Vol. 45 (2004), No. 2, 55--58

Persistent URL: http://dml.cz/dmlcz/702097

# Discontinuity Points of Exactly $k$-to-one Functions

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*Received 15. March 2004*

For a natural number $k \geq 1$ and a topological space $X$ the following question is considered. If $F \geq X$ is an infinite $F_\sigma$-set that contains no point isolated in $X$, does there exist an exactly $k$-to-one functin $f : X \xrightarrow{\text{onto}} X$ whose set of all discontinuity points is $F$? The answer is given for $k = 1$ if $X$ is a separable metrizable space, and for $k \geq 1$ if $X = [0,1]$.

A function is (*exactly*) *k-to-one* if the preimage of every point has exactly $k$ elements. O. G. Harrold [1] showed that *no two-to-one continuous function can be defined on the interval* $[0,1]$. Jo W. Heath [2] proved that *no two-to-one function from* $[0,1]$ *into a Hausdorff space has a finite number of discontinuity points.* For each natural number $k \geq 3$ there is a $k$-to-one continuous function from $[0,1]$ onto the circle, see [1]. H. Katsuura and K. R. Kellum [4] showed that *for* $k \geq 2$ *there is no k-to-one function* $f : [0,1] \xrightarrow{\text{onto}} [0,1]$ *with finitely many discontinuity points.* Several other authors have considered $k$-to-one functions, see a survey [3] by Heath.

If $f : X \to Y$ is a function into a metrizable space $Y$, then the set of all discontinuity points of $f$ is an $F_\sigma$-set. For each $k \geq 1$ we prove that *every infinite* $F_\sigma$*-set* $F \subseteq [0,1]$ *is the set of all discontinuity points of a certain k-to-one function* $f : [0,1] \xrightarrow{\text{onto}} [0,1]$. In the crucial case of bijections (involutions) between infinite countable sets, we develop an idea of S. S. Kim and Sz. Plewik [5]. In fact, we prove that *if* $X$ *is a separable metrizable space and* $F \subseteq X$ *is an infinite* $F_\sigma$*-set*

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2000 *Mathematics Subject Classification:* Primary 26A15, Secondary 54C10, 54C30.
*Key words and phrases:* discontinuity, exactly $k$-to-one function, $F_\sigma$-set, involution.

*which has no point isolated in $X$, then there exists an involution $\varphi : X \to X$ such that $F$ consists of all discontinuity points of $\varphi$.*

Recall that a function $\varphi : X \to X$ is an *involution* if $\varphi \circ \varphi = \mathrm{id}_X$. A metric space $X$ is *totally bounded* if for every $\varepsilon > 0$ there exists a finite cover $\mathscr{U}$ of $X$ which consists of open sets of diameters less thatn $\varepsilon$. Every separable metrizable space has a totally bounded metric.

**Lemma 1.** *If $(P, \varrho)$ is a countable, infinite, and totally bounded metric space, then $P$ can be arranged into a one-to-one sequence $x_0$, $x_1$, $x_2$, ... such that*

$$\lim_{n\to\infty} \varrho(x_{2n}, x_{2n+1}) = 0.$$

*Proof.* Let $\mathscr{U}_0 = \{P\}$. For each $n > 0$ fix a finite open cover $\mathscr{U}_n$ of $P$ which consists of sets of diameter less than $1/n$. Let $P = \{p_0, p_1, p_2, \ldots\}$. Put $x_0 = p_0$ and choose $x_1 \in P \setminus \{x_0\}$. Suppose that the elements $x_0, x_1, \ldots, x_{2n-1}$ have been defined. Let $x_{2n} = p_i$, where $p_i \in P \setminus \{x_0, x_1, \ldots, x_{2n-1}\}$ and $i$ is the least possible index. Take the greatest index $k \leq n$ such that there exists an infinite set $I \in \mathscr{U}_k$ with $x_{2n} \in I$. Then choose $x_{2n+1} \in I \setminus \{x_0, x_1, \ldots, x_{2n}\}$. By induction the one-to-one sequence $x_0, x_1, x_2, \ldots$ has been defined. Clearly $P = \{x_0, x_1, x_2, \ldots\}$.

Fix a natural number $m > 0$. Take $l \geq m$ such that

$$\bigcup\{I \in \mathscr{U}_m : I \text{ is finite}\} \subseteq \{p_0, p_1, \ldots, p_l\}.$$

If $n > l$, then $x_{2n} \in \{p_{l+1}, p_{l+2}, \ldots\}$, and $x_{2n}$ belongs to an infinite set $I \in \mathscr{U}_m$. Hence $x_{2n}$ and $x_{2n+1}$ belong to a set $J \in \mathscr{U}_k$, where $m \leq k \leq n$. We have $\varrho(x_{2n}, x_{2n+1}) \leq \operatorname{diam} J < 1/k \leq 1/m$. Therefore, $\lim_{n\to\infty} \varrho(x_{2n}, x_{2n+1}) = 0$ since $m$ could be taken arbitrarily. ■

**Corollary.** (cf. [5]). *Suppose that $X$ is a separable metrizable space, and $P \subseteq X$ is infinite and countable. Then, there exists an involution $\varphi : X \to X$ such that $P = \{x \in X : \varphi(x) \neq x\}$ and $\lim_{t\to x} \varphi(t) = x$ for any non-isolated point $x \in X$.*

*Proof.* Since $X$ has a totally bounded metric, we can arrange $P$ into a sequence $x_0, x_1, x_2, \ldots$ as in Lemma 1. Put $\varphi(x_{2n}) = x_{2n+1}$ and $\varphi(x_{2n+1}) = x_{2n}$ for each $n$. Extend this function by putting $\varphi(x) = x$ for every $x \in X \setminus P$. ■

**Lemma 2.** *If $(P, \varrho)$ is a countable, dense-in-itself, and totally bounded metric space, then for every $\varepsilon > 0$ there is $\delta > 0$ and an involution $\varphi : P \to P$ such that $\delta < \varrho(p, \varphi(p)) < \varepsilon$ for any $p \in P$.*

*Proof.* Cover $P$ by non-empty open subsets $P_1, P_2, \ldots, P_n$ with diameters less than $\varepsilon > 0$. Put

$$\delta = \frac{1}{3} \min\{\operatorname{diam} P_k : k = 1, \ldots, n\} > 0.$$

By induction, for every $p \in P$ choose $\varphi(p)$ such that $p$, $\varphi(p) \in P_k$ for a certain $k$, and $\delta < \varrho(p, \varphi(p))$. Put $\varphi(\varphi(p)) = p$. ■

**Theorem 1.** *Suppose that $X$ is a separable metrizable space, and $F \subseteq X$ is an infinite $F_\sigma$-set which has no point isolated in $X$. Then, there exists an involution $\varphi : X \to X$ whose set of all discontinuity points is $F$. Moreover, if $x \in X \setminus F$, then $\varphi(x) = x$, and the set $\{x \in X : x \neq \varphi(x)\}$ is countable.*

*Proof.* If $F$ is countable, then Corollary works. In the other case, let $G_0, G_1, \ldots$ be pairwise disjoint sets such that $G_0 \cup G_1 \cup \ldots = F$ and each union $G_0 \cup G_1 \cup \ldots \cup G_n$ is closed. As $F$ is uncountable, it contains a convergent sequence, and hence, we can assume that $G_{00}$ is closed, scattered, and infinite. For $n > 0$ divide $G_n$ into the scattered part and the dense-in-itself part. The dense-in-itself part of $G_n$ denote by $H_n$. Let $P$ be the union of the scattered parts of all $G_n$. The set $P \supseteq G_0$ is countable and infinite.

Since $X$ has a totally bounded metric $\varrho$, Lemma 1 works in the same way as in Corollary. There exists an involution $\psi : P \to P$ such that $\psi(x) \neq x$ for every $x \in P$ and $\lim_{t \to y} \psi(t) = 0$ for any cluster point $y$ of $P$.

If $H_n$ is non-empty, find a countable dense subset $Q_n \subset H_n$ such that $H_n \setminus Q_n$ is dense in $H_n$, too. By Lemma 2 there is $\delta_n > 0$ and an involution $\zeta_n : Q_n \to Q_n$ such that $\delta_n < \varrho(x, \zeta_n(x)) < 1/n$ for every $x \in Q_n$. Put: $\varphi(x) = \psi(x)$ if $x \in P$; $\varphi(x) = \zeta_n(x)$ if $x \in Q_n$; and $\varphi(x) = x$ if $x \in X \setminus Q_1 \cup Q_2 \cup \ldots \cup P$.

If $x \in X \setminus F$ and $\lim_{n \to \infty} x_n = x$, then $x = \varphi(x) = \lim_{n \to \infty} \varphi(x_n)$, and hence $\varphi$ is continuous at $x$.

Any point of $G_n \setminus P$ is a cluster point of the set $H_n \setminus Q_n \subseteq \{x \in X : \varphi(x) = x\}$, and is a cluster point of $Q_n$. But $\varphi$ moves points of $Q_n$ at distance greater than $\delta_n$, and hence, it can be continuous at no point of $F \setminus P$.

Finally, no point of $P$ is isolated in $X$, and $\varphi$ moves each point of $P$. Hence $\varphi$ is discontinuous at any point of $P$. ■

**Theorem 2.** *If $k > 1$ and $F \subseteq [0,1]$ is an infinite $F_\sigma$-set, then there exists a k-to-one function $f : [0,1] \xrightarrow{\text{onto}} [0,1]$ whose set of all discontinuity points is $F$.*

*Proof.* Let $0 = b_0 < b_1 < \ldots < b_k = 1$ be such that each interval $(b_i{-}1, b_i)$ contains at least two points in $F$. Fix an interval $(b_{j-1}, b_j)$ which contains a convergent one-to-one sequence $\{x_0, x_1, x_2, \ldots\} \subset F$. Any $A_i = F \cap (b_{i-1}, b_i) \setminus \{x_0, x_1, x_2, \ldots\}$ is an $F_\sigma$-set. If $A_i$ is infinite, choose an involution $\varphi_i : : [b_{i-1}, b_i] \xrightarrow{\text{onto}} [b_{i-1}, b_i]$ whose set of all discontinuity points is $A_i$, and which is the identity on $[b_{i-1}, b_i] \setminus A_i$ (use Theorem 1). If $A_i$ is finite, let $\varphi_i : [b_{i-1}, b_i] \xrightarrow{\text{onto}} [b_{i-1}, b_i]$ be a bijection such that $\varphi_i(x) \neq x$ for every $x \in A_i$. Since $b_i \notin A_i \cup A_{i+1}$, we have $\varphi_i(b_i) = b_i = \varphi_{i+1}(b_i)$ for $0 < i < k$. Hence $\varphi = \varphi_1 \cup \varphi_2 \cup \ldots \cup \varphi_k$ is a bijection from $[0,1]$ onto $[0,1]$.

Consider the continuous function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g$ maps each interval $[b_{i-1}, b_i]$ linearly onto $[0,1]$. We shall define a function

$\alpha : [0,1] \xrightarrow{\text{onto}} [0,1]$ so that the desired $k$-to-one functions is $f = g \circ \varphi \circ \alpha$.

Denote $B = F \cap \{b_0, b_1 \dots, b_k\} \cap \{x_0, x_1 \dots\} = \{b_{i_0}, b_{i_1}, \dots, b_{i_m}, x_0, x_1, \dots\}$. If $F \cap \cap \{b_0, b_1, \dots, b_k\} = \emptyset$, take $m = -1$. Let $\alpha$ be the identity on $[0,1] \setminus B$. Put: $\alpha(b_{i_j}) = x_j$ and $\alpha(x_j) = b_{i_j}$ if $0 \leq j \leq m$; and $\alpha(x_{j+k\ 1}) = x_j$ if $j \geq m + 1$. Finally, choose $\alpha(x_{m+1}), \dots, \alpha(x_{m+k-1}) \in \{b_0, b_1\}$ so that the preimages $f\ ^{1}(0)$ and $f\ ^{1}(1)$ have exactly $k$ elements.

By the definition, the composition $f = g \circ \varphi \circ \alpha$ is a $k$-to-one function. If $x \in B$, then $x \neq \alpha(x) \in B$. Therefore, the composition $\varphi \circ \alpha$ is continuous at no point of $B$, and hence, $f$ is discontinuous at any point of $B$. The function $\varphi$ is continuous at no point of $A_1 \cup \dots \cup A_k$, and hence, $f$ is discontinuous at any point of $A_1 \cup \dots \cup A_k$. Thus, $A_1 \cup \dots \cup A_k \cup B = F$ consists of all discontinuity point of $f$. ■

## References


[1] Harrold, O. G., *The non-existence of a certain type of continuous transformation*, Duke Math. J. 5 (1939), 789 – 793.

[2] Heath, J. W., *Every exactly 2-to-1 function on the reals has an infinite set of discontinuites*, Proc. Amer. Math. Soc. 98 (1986), 369 – 373.

[3] Heath, J. W., *Exactly k-to-*1 *maps: From pathological functions with finitely many discontinuities to well-behaved covering maps*, [in:] *Continua. With the Houston Problem book*, 89 – 102, Lecture Notes In Pure and Appl. Math. 170, Dekker, New York, 1995.

[4] Katsuura, H. and Kellum, K. R., *k-to-1 function on an arc*, Proc. Amer. Math. Soc. 101 (1987), 629 – 633.

[5] Kim S. S. and Plewik, Sz., *Discontinuity and involutions on countable sets,* Annales Mathematicae Silesianae 17 (2003), 7 – 8.