Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 44 (2003), No. 2, 97--103

Persistent URL: http://dml.cz/dmlcz/702092

On Bijective Isometries

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Received 11. March 2003

We give a local version of a theorem of Wobst on affinity of surjective isometries between real linear metric spaces. We also give an extension to real linear metric spaces of a result due to Vogt that every surjective continuous equality of distance preserving map between real normed spaces is affine. A basic tool here is the study of bijective isometries in dissimilarity spaces.

1. Introduction

We shall deal with bijective isometries in metric spaces and, more general, in dissimilarity spaces. In particular, we shall deal with the following known conjecture.

Conjecture (H). *A surjective isometry between real F^*-spaces X and Y is affine.*

Recall some definitions.

By a *dissimilarity space* (X, d) one means a nonempty set X endowed with a nonnegative function $d : X \times X \to \mathbf{R}$ satisfying the following conditions:

(i) $d(x, x) = 0$,

(ii) $d(x, y) = d(y, x)$.

Such a function $d(x, y)$ is called a *dissimilarity*; it is called a *definite dissimilarity* if in addition

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1991 *Mathematics Subject Classification.* [2000] Primary 46A16.

Key words and phrases. dissimilarity, bijective isometry, map preserving equality of distance, F^*-spaces, affinity.

The author was supported by the Edmund Landau Center for Research in Mathematical analysis and Related Areas, sponsored by the Minerva Foundation (Germany).

(iii) $d(x, y) = 0 \Rightarrow x = y$.

A definite dissimilarity satisfying the *triangle inequality*

(iv) $d(x, y) \leq d(x, z) + d(z, y)$

is called a *metric*, and then $X = (X, d)$ is called a *metric space*.

Let $X = (X, d)$ and $Y = (Y, \rho)$ be dissimilarity spaces. Following Vogt [Vo], we shall say that a map $f : X \to Y$ *preserves equality of distance* if there exists a function $p: [0, \infty) \to [0, \infty)$ such that

$$\rho(f(x), f(y)) = p(d(x, y))$$

for every x and y in X. The function p is called the *gauge function* for f. If $p = id$ (the identity), then f is called an *isometry*.

By an F^*-*space* (X, d) one means a linear space X endowed with a translation invariant metric d (i.e., $d(x + y, y) = d(x, 0)$ for all $x, y \in X$) such that the operations of addition and multiplication are continuous, (i.e., $t_n x_n \to tx$ provided $t_n \to t$ in $\mathbf{R}$ and $x_n \to x$ in (X, d)). The functional $\|x\| = d(x, 0)$ is called *F-norm*.

The example of the function $t \mapsto (t, |t|)$ from $\mathbf{R}$ to l^2_∞ shows that in general surjectivity is essential in (H). (But it can be dropped if Y is a strictly convex normed space, that is, a sphere does not contain a line segment.) Note also that (H) is not valid for complex spaces (just consider complex conjugation on $\mathbf{C}$).

Although it is unknown whether (H) holds in general, it has been proved under some additional assumptions; I mention here only some of them:

(i) The classical Mazur–Ulam theorem [MU] asserts that it holds for linear normed spaces. Moreover, John [J] showed that any open local isometry which maps an open connected subset of a real normed space X onto an open subset of another real normed space Y is the restriction of an affine isometry of X onto Y.

(ii) Charzyński [C] proved (H) for F^*-spaces of the same finite dimension. (Note that in this case, any isometry between X and Y is surjective by the invariance of domains.)

(iii) Wobst [W] proved (H) under the assumption that there are $A > 0$ and a non-decreasing function γ such that $\gamma(t) > 1$ for every $t > 0$ and $\|2x\| \geq \gamma(\|x\|)\,\|x\|$ for every $x \in X$ with $\|x\| \leq A$.

Recently Väisälä [Va] proposed a new elegant proof for the Mazur–Ulam theorem. His proof is based on ideas of Vogt [Vo] and makes use of reflections in points.

In the present paper we use the same method to give some generalizations of the Mazur–Ulam theorem, including a local version of a generalized Wobst theorem (see Theorem 6) and an extension to F^*-spaces of a result of Vogt [Vo] on affinity of a surjective continuous equality of distance preserving map between real normed spaces (see Theorem 8).

We use standard notation. As usual (x, y) and $[x, y]$ denote the open and closed straight line segments joining the points x and y in an F^*-space. The ball and the sphere with center z and radius r in a dissimilarity space X are denoted by $B(z, r)$ and $S(z, r)$, that is, $B(z, r) = \{x : d(x, z) \leq r\}$ and $S(z, r) = \{x : d(x, z) = r\}$. When X has a linear structure, we abbreviate $B(0, r) = B(r)$ (or $B_X(r)$ when we need to specify the space).

Let X be an F^*-space. For $z \in X$, the *reflection of X in z* is the map $\psi(x) = 2z - x : X \to X$. Clearly, ψ is an isometric involution (i.e., $\psi^2 = id$) with a unique fixed point z, and

$$(1) \qquad \psi(x) - z = z - x, \qquad \psi(x) - x = 2(z - x)$$

for every $x \in X$.

2. Dissimilarity spaces

We start our investigation with some statements about bijective isometries in dissimilarity spaces (we want to emphasize that if the dissimilarity of a space is definite then any isometry is injective, so any surjective isometry is bijective). Throughout this section $X = (X, d)$ and $Y = (Y, \rho)$ denote dissimilarity spaces which are not singleton.

A point $z \in X$ will be called a *dissimilarity center* (d.c.) of X if $X \subseteq B(z, R)$ for some $R > 0$ and there is a family of bijective isometries $\{\psi_x : X \to X \mid x \in X\}$ such that for every $\emptyset \neq S \subseteq X$ with $S \neq \{z\}$

$$(2) \qquad \sup_{x \in S} d(\psi_x(x), x) > \sup_{x \in S} d(x, z)$$

The following basic lemma generalizes [Vo, Theorem 1.2] of Vogt.

Lemma 1. *Let z be a d.c. of X. Then $f(x) = z$ for each bijective isometry f of X.*

Proof. We use here a method due to Väisälä [Va].

Set $S = \{g(z) : g$ is a bijective isometry of $X\}$. Suppose that $S \neq \{z\}$. Let $\{\psi_x\}_{x \in X}$ be bijective isometries associated with z as a d.c. of X. By the definitions, there is $y \in S$ such that $d(\psi_y(y), y) > \sup_{x \in S} d(x, z)$. Then there is a bijective isometry g with $g(z) = y$. Note also that $g^{-1}(\psi_{g(z)} g(z)) \in S$. Thus

$$d(\psi_y(y), y) = d(\psi_{g(z)} g(z), g(z)) = d(g^{-1}(\psi_{g(z)} g(z)), z) \leq \sup_{x \in S} d(x, z),$$

a contradiction. Thus $S = \{z\}$, which was to be proved. □

Note that z is a unique common fixed point of all $\{\psi_x\}$. Indeed, if z' is another one, take $S = \{z'\}$. Then by (2), $d(z, z') < d(\psi_{z'}(z'), z') = 0$, a contradiction.

Lemma 2. *Any dissimilarity space has at most one d.c.*

Proof. Indeed, assume that z and z' are d.c. Let $\{\psi'_x\}$ be bijective isometries corresponding to z'. By Lemma 1, z is fixed by all ψ'_x. But z' is the only such a point, hence $z' = z$. □

Lemma 3. *Let $f : X \to Y$ be a bijective map preserving equality of distance with a gauge function p which strictly increases. If X has a d.c. z, then $f(z)$ is a d.c. of Y.*

Proof. Choose $R > 0$ such that $X \subseteq B(z, R)$. Then $Y \subseteq B(f(z), p(R))$.

Let $\{\psi_x\}_{x \in X}$ be bijective isometries associated with z, and set $\psi'_y = f\psi_{f^{-1}(y)}f^{-1}$ for $y \in Y$. Then for every $y, u, v \in Y$

$$\rho(\psi'_y(u), \psi'_y(v)) = p(d(f^{-1}(u), f^{-1}(v))) = \rho(u, v),$$

i.e., each ψ'_y is a bijective isometry on Y.

Let $\emptyset \neq S' \subseteq Y$ with $S \neq \{f(z)\}$. Set $S = f^{-1}(S')$ and $c = \sup_{x \in S} d(x, z)$. Since p strictly increases, we have

$$\begin{aligned}\sup_{y \in S'} \rho(\psi'_y(y), y) &= \sup_{y \in S'} p(d(\psi_{f^{-1}(y)}f^{-1}(y), f^{-1}(y))) = \sup_{x \in S} p(d(\psi_x(x), x)) \\ &> p(c) \geq \sup_{x \in S} p(d(x, z)) = \sup_{y \in S'} \rho(y, f(z)),\end{aligned}$$

which completes the proof. □

For the convenience, we also introduce a notion of a metric center of a pair of points in a metric space (not necessarily bounded).

Let X be a metric space, and let $a, b \in X$. A point $z \in X$ will be called a *metric center* of a, b if there is a surjective isometry $\psi : X \to X$ such that $\psi(a) = b$, $\psi(b) = a$ and for every $S \subseteq X$ with $0 < \sup_{x \in S} d(x, z) < \infty$

$$\sup_{x \in S} d(\psi(x), x) > \sup_{x \in S} d(x, z). \tag{3}$$

Note that if X is bounded, then z is a dissimilarity center of X (with $\psi_x = \psi$).

Lemma 4. *Let X be a metric space, and let z be a m.c. of $a, b \in X$. Then*
(i) z is a unique metric center of a, b.
(ii) z is a unique fixed point of ψ.
(iii) Let Y be another metric space. If $f : X \to Y$ is a surjective map preserving equality of distance with a gauge function p which strictly increases, then $f(z)$ is a m.c. of $f(a), f(b)$.

Proof. (i) Let ψ be a surjective isometry associated with z as a m.c. of a, b, and assume that z' is another m.c. of a, b with an associated map ψ'. Set $d = \max\{d(a,z), d(a, z')\}$ and $Z = B(a, d) \cup B(b, d)$. Then $z, z' \in Z$, and ψ and ψ' are bijective isometries on Z. Thus, z and z' are d.c. of Z, and hence $z' = z$ by Lemma 2.

(ii) Since z is a d.c. of Z, it follows from Lemma 1 that z is fixed by ψ. Again, if z' is another fixed point, set $S = \{z'\}$. Then by (3), $d(z, z') < d(\psi(z'), z') = 0$, a contradiction.

(iii) The proof follows the same path as the proof of Lemma 3 with $\psi' = f\psi f^{-1}$ and with the observation that $\psi'(f(a)) = f(b)$ and $\psi'(f(b)) = f(a)$. □

3. F^*-spaces

Throughout this section X and Y denote real F^*-spaces.

Let S be a convex subset of X. Then a continuous map $f : S \to Y$ is affine iff it preserves midpoints, i.e.,

$$f\left(\frac{a+b}{2}\right) = \frac{f(a) + f(b)}{2}$$

for all $a, b \in S$.

Since in a normed space $z = (a + b)/2$ is a m.c. of the points a, b (with the reflection in z as an associated isometry), the Mazur-Ulam theorem follows from Lemma 4.

To simplify formulations we introduce some additional notation.

Given an F^*-space X, set for every $r > 0$

$$\alpha(r) = \sup\{\|(x + y)/2\| : x, y \in B(r)\}.$$

Clearly, the function α does not decrease and $\alpha(r) \geq r$. Also, $\lim_{r\to 0} \alpha(r) = 0$. Indeed, otherwise there are $c > 0$ and sequences $\{x_n\}, \{y_n\} \subset X$ such that $\|x_n\|, \|y_n\| < 1/n$ and $\|(x_n + y_n)/2\| > c$. But this contradicts the continuity of addition, since $\|x_n + y_n\| < 2/n$.

We say that X has *property (RS)* if there is $R > 0$ such that for every nonempty set $S \subseteq B(R)\backslash\{0\}$

$$\sup_{x\in S} \|2x\| > \sup_{x \in S} \|x\|.$$

Lemma 5. *Let X have property (RS) and $r > 0$. Let $f : B_X(u, 2\alpha(r)) \to B_Y(f(u), p(2\alpha(r)))$ be a continuous surjective map preserving equality of distance with an injective gauge function p. Then f is affine on a segment $[a, b]$ provided $[a, b] \subseteq B_X(u, r)$.*

Proof. Set $c = \|(a - b)/2\|$, $z = (a + b)/2$ and $Z = S(a, c) \cup S(b, c) \cup \{a\} \cup \{b\}$. Since $\|a - u\|$, $\|b - u\| \leq r$, then $c \leq \alpha(r)$. Consequently, $z \in Z \subseteq B(z, 2c) \cap B(u, 2\alpha(r))$.

First, we claim that if $2c \leq R$, then

$$f\left(\frac{a+b}{2}\right) = \frac{f(a) + f(b)}{2} \tag{4}$$

By translation, we can assume that $b = -a$ (that is $z = 0$). Set $\psi = -id$ on X. Then $Z \subseteq B(R)$, and ψ is an isometric involution on Z. Let $\emptyset \neq S \subseteq Z \setminus \{0\}$.Then by the assumptions,

$$\sup_{x \in S} \|\psi(x) - x\| = \sup_{x \in S} \|2x\| > \sup_{x \in S} \|x\| .$$

Thus, the origin is a d.c. of Z.

Set $\psi' = -id + f(a) + f(-a)$ (on Y). Clearly, ψ' is an isometric involution on $f(Z) = S(f(a), p(c)) \cup S(f(-a), p(c)) \cup \{f(a)\} \cup \{f(-a)\}$. Since p is injective, then so is f. Set $h = f^{-1}\psi' f$ on Z. Then for all $x, y \in Z$, $\|h(x) - h(y)\| = p^{-1}(\|f(x) - f(y)\|) = \|x - y\|$, i.e., h is a bijective isometry on Z. By Lemma 1, $h(0) = 0$, i.e., $f(a) + f(-a) - f(0) = f(0)$, and (4) follows.

Since the operations of addition and multiplication by scalar are continuous, there is a $\delta > 0$ such that $\|\lambda(a - b)\| < R/2$ whenever $|\lambda| < \delta$. Consider the function $g(t) = f(ta + (1 - t)\, b)$ on $[0, 1]$. By the claim above

$$g\left(\frac{s + t}{2}\right) = \frac{g(s) + g(t)}{2}$$

whenever $|s - t| < 2\delta$. Thus, g is affine on every sub-interval with the length less than 2δ. Covering $(0, 1)$ by such small intervals completes the proof. □

Theorem 6. *Let X have property (RS). Let f be a map from an open connected subset Ω of X onto an open subset of Y, which satisfies the following condition:*

– *every point $x \in \Omega$ has an open neighborhood V such that $f(V)$ is open and f preserves equality of distance in V with a strictly increasing gauge function.*

Then f is the restriction of an affine uniform homeomorphism of X onto Y.

Proof. For the matter of the proof we introduce one more function which can be roughly considered as an inverse of the function α. Namely, we set for every $r > 0$

$$\beta(r) = \sup\left\{s \geq 0 : \bigcup_{0 \leq t \leq 1} (tB(s) + (1 - t)\, B(s)) \subseteq B(r)\right\}.$$

It is easy that for any ball $B(r)$ there is a balanced neighborhood of zero V such that $V + V \subset B(r)$. (Recall that a set V is called *balanced* if $tV \subset V$ whenever $|t| \leq 1$.) Therefore, the function β is positive for every positive r.

Let f maps $B_X(u, 2\alpha(r))$ with a strictly increasing gauge function p. Then f is uniformly homeomorphic in $B_X(u, 2\alpha(r))$ and, by Lemma 5, f is affine on $[a, b]$ provided $a, b \in B_X(u, \beta(r))$. Since Ω can be covered by small balls so that f is affine on each line segment joining two points of the same ball, and since two affine maps that agree on a ball are the restriction of the same affine map, it follows from the connectedness of Ω that f is the restriction of an affine map h on X. The map $L(x) = h(x + u) - h(u)$ is linear on X and homeomorphically maps $B_X(2\alpha(r))$ onto $B_Y(p(2\alpha(r)))$. Thus, it is a desired homeomorphism. □

Now the Wobst theorem follows from Theorem 6 and the next simple assertion.

Lemma 7. *Let X be F^*-space. Suppose that there are $A > 0$ and non-decreasing function γ such that $\gamma(t) > 1$ for every $t > 0$ and $\|2x\| \geq \gamma(\|x\|)\,\|x\|$ for every $x \in X$ with $|x\| \leq A$. Then X has property (RS).*

Proof. Let $\emptyset \neq S \subseteq B(A)\setminus\{0\}$ and put $s = \sup_{x\in S}\|x\|$. For every $0 < \varepsilon < 1/2$,

$$\|2x_\varepsilon\| \geq \gamma(\|x_\varepsilon\|)\,\|x_\varepsilon\| \geq \gamma(s/2)\,\|x_\varepsilon\|.$$

Hence

$$\sup_{x\in S}\|2x\| \geq \gamma(s/2)\,s > s. \qquad \square$$

Unfortunately, I do not know is there a space with (RS) which is not covered by the Wobst theorem.

The next statement generalized [Vo, Theorem 1.3(i)] of Vogt.

Theorem 8. *Suppose that for every nonempty bounded set $S \subset Y\setminus\{0\}$*

$$\sup_{y\in S}\|2y\| > \sup_{y\in S}\|y\|.$$

If $f : X \to Y$ is a continuous surjective map preserving equality of distance, then f is affine.

Note that no injectivity of the gauge function is required here.

The proof is the same as the proof of [Vo, Theorem 1.3(i)] with use of Lemma 1 instead of [Vo, Theorem 1.2]. We refer the interested reader to this paper.

Acknowledgment. I express my gratitude to Professor Y. Benyamini for fruitful discussions. I am also grateful to the referee who helped me to improve the presentation of the paper.

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