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Klaas Pieter Hart; B. J. van der Steeg

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On the Maćkowiak-Tymchatyn Theorem

KLAAS PIETER HART and B. J. VAN DER STEEG

Delft

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In this paper we give new proofs of the theorem of Maćkowiak and Tymchatyn that every metric continuum is a weakly-confluent image of some one-dimensional hereditarily indecomposable continuum of countable weight. One is a model-theoretic argument; the other a topological one. Both proofs make essential use of two (topological) lemmas.

1. Introduction

In [5] Maćkowiak and Tymchatyn proved that every metric continnum is the continuous image of a one-dimensional hereditarily indecomposable continuum by a weakly confluent map. In [3] this result was extended to general continua, with two proofs, one topological and one model-theoretic. Both proofs made essential use of the metric result.

The original purpose of this paper was to (re)prove the metric case by model-theoretic means. After we found this proof we realized that it could be combined with any standard proof of the completeness theorem of first-order logic (see e.g., Hodges [4], 6.1]) to produce an inverse-limit proof of the general form of the Maćkowiak-Tymchatyn result. We present both proofs. The model-theoretic argument occupies section 5, and the inverse-limit approach appears in section 4.

We want to take this opportunity to point out some connections with work of Bankston [1], who dualized the model-theoretic notions of existentially closed

Faculty on Information Technology and Systems, TU Delft, Postbus 5031, 2600 GA Delft, The Netherlands, e-mail: k.p.hart@its.tudelft.nl; e-mail: b.j.vandersteeg@its.tudelft.nl

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structures and existential maps to that of co-existential maps are weakly confluent, that co-existentially closed continua are one-dimensional and hereditarily indecomposable, and that every continuum is the continuous image of a co-existentially closed one. The map can in general not be chosen co-existential, because co-existential maps preserve indecomposability and do not raise dimension.

The paper is put together in such a way that readers who are only interested in the topological (model-theoretic) proof can simply ignore section 5 (section 4 respectively) without loss of continuity.

2. Preliminaries

A continuum is *decomposable* if it can be written as a union of two proper subcontinua, it is called *indecomposable* if this is not the case. We call a continuum *hereditarily indecomposable* if every subcontinuum is indecomposable. This is equivalent to saying that of every two subcontinua that meet, one is contained in the other. As in [3] we can extend this notion for arbitrary compact Hausdorff spaces. So a compact Hausdorff space is hereditarily indecomposable if for every two subcontinua that meet, one is contained in the other. We call a continuous mapping between two continua *weakly confluent* if every subcontinuum in the range is the image of a subcontinuum in the domain.

Theorem 1 (Maćkowiak and Tymchatyn [5]). *Every metric continuum is a weakly confluent image of some one-dimensional hereditarily indecomposable metric continuum.*

For our purposes it is necessary to have a characterization of hereditary indecomposability that does not mention continua.

Theorem 2 (Krasinkiewicz and Minc). *A compact Hausdorff space is hereditarily indecomposable if and only if it is crooked between every pair of disjoint closed nonempty subsets.*

Which the authors translated in [3] into terms of closed sets only as follows.

Theorem 3. [3] *A compact Hausdorff space X is hereditarily indecomposable if and only if whenever four closed sets C, D, F and G in X are given such that $C \cap D = C \cap G = F \cap D = \emptyset$ one can write X as the union of three closed sets X_0, X_1 and X_2 such that $C \subset X_0$, $D \subset X_2$, $X_0 \cap X_1 \cap F = \emptyset$, $X_0 \cap X_2 = \emptyset$ and $X_1 \cap X_2 \cap G = \emptyset$.*

Further on in the paper we will make use of the following characterization of the covering dimension dim for normal spaces.

Lemma 1. *A normal space X satisfies the condition* $\dim(X) \leq n$ *if and only if for every $(n+2)$-element family $\{B_1, B_2, \ldots, B_{n+2}\}$ closed subsets of the space X*

satisfying $\bigcap_{i=1}^{n+2} B_i = \emptyset$ *there exists a closed cover* $\{F_1, F_2, \ldots, F_{n+2}\}$ *of the space* X *such that* $\bigcap_{i=1}^{n+2} F_i = \emptyset$, *and* $B_i \subset F_i$ *for all* i.

3. Two main lemmas

The two lemmas in this section stand at the basic of the topological as well as the model-theoretic proof in section 4 and section 5 respectively.

Lemma 2. *If* X *is a continuum and* a, b *and* c *are nonempty closed subsets of* X *with empty intersection then there exist a continuum* Y *and a monotone closed onto map* $\phi : Y \to X$ *such that* $w(X) = w(Y)$ *and* Y *has a closed cover* $\{A, B, C\}$ *with the property that* $\phi^{-1}[a] \subset A$, $\phi^{-1}[b] \subset B$, $\phi^{-1}[c] \subset C$ *and* $A \cap B \cap C = \emptyset$.

Proof. We apply normality to find a partition of unity $\{\kappa_a, \kappa_b, \kappa_c\}$ subordinate to $\{X\backslash a, X\backslash b, X\backslash c\}$, i.e. the support of κ_a is a subset of $X\backslash a$, etc. Define the function $f : X \to \mathbb{R}^3$ by $f(x) = (\kappa_a(x), \kappa_b(x), \kappa_c(x))$. The function f maps the space X into the triangle $T = \{(t_1, t_2, t_3) \in \mathbb{R}^3 : t_1, t_2, t_3 \geq 0$ and $t_1 + t_2 + t_3 = 1\}$. The resulting embedding of X into $X \times T$ defined by $x \mapsto (x, f(x))$, will be denoted by g.

Now consider the space $\partial T \times [0, 1]$, where $\partial T = T\backslash \text{int}(T)$ in $\mathbb{R}^3$. Let h be the map from $\partial T \times [0, 1]$ onto T defined by

$$h((x, t)) = x(1 - t) + t\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

The map h restricted to $\partial T \times [0, 1)$ is a homeomorphism between $\partial T \times [0, 1)$ and $T\backslash\{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}$.

We define $Y \subset X \times (\partial T \times [0, 1])$ by $Y = (\text{id} \times h)^{-1}\,[g[X]]$. And let $\phi : Y \to X$ be the (onto) map $g^{-1} \circ (\text{id} \times h)$. As the inverse images of points $(x, (t_1, t_2, t_3))$ under the map $\text{id} \times h$ are points for $(x, (t_1, t_2, t_3))$ in $X \times T$ with $(t_1, t_2, t_3) \neq (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and equal to $\{x\} \times \partial T \times \{1\}$ for those $(x, (t_1, t_2, t_3))$ in $X \times T$ with $(t_1, t_2, t_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we find that the map $\text{id} \times h : X \times (\partial T \times [0, 1]) \to X \times T$ is monotone. Furthermore it is also closed.

Let p be the line segment between $(0, 1, 0)$ and $(0, 0, 1)$, q the line segment between $(1, 0, 0)$ and $(0, 0, 1)$ and r the line segment between $(1, 0, 0)$ and $(0, 1, 0)$. The sets $A = Y \cap (X \times (p \times [0, 1]))$, $B = Y \cap (X \times (q \times [0, 1]))$ and $C = Y \cap (X \times (r \times [0, 1]))$ form a closed cover of Y such that $\phi^{-1}[a] \subset A$, $\phi^{-1}[b] \subset B$, $\phi^{-1}[c] \subset C$, and $A \cap B \cap C = \emptyset$. As it is easily seen that Y and X have the same weight, we have proven the lemma. □

Lemma 3. *If* X *is a continuum and* a, b, c *and* d *are nonempty closed subsets of* X *such that* $a \cap b = a \cap d = b \cap c = \emptyset$ *then there exist a continuum* Y *and a weakly confluent onto map* $\psi : Y \to X$ *such that* $w(X) = w(Y)$ *and* Y *has a closed cover* $\{U, V, W\}$ *with the property that* $\psi^{-1}(a) \subset U$, $\psi^{-1}(b) \subset W$ *and* $U \cap V \cap \psi^{-1}(c) = U \cap W = V \cap W \cap \psi^{-1}(d) = \emptyset$.

Proof. We are going to use an idea from [3]. Let a, b, c and d be nonempty closed subsets of X with the property stated in the lemma. With the aid of Urysohn's lemma we can find a continuous function $f : X \to [0, 1]$ such that $f[a] \subset \{0\}, f[b] \subset \{1\}, f[c] \subset [0, \frac{1}{2}]$ and $f[d] \subset [\frac{1}{2}, 1]$.

Let P denote the (closed and connected) subset of $[0, 1] \times [0, 1]$ given by

$$P = \left\{\frac{1}{4}\right\} \times \left[0, \frac{2}{3}\right] \cup \left[\frac{1}{4}, \frac{1}{2}\right] \times \left\{\frac{2}{3}\right\} \cup \left\{\frac{1}{2}\right\} \times \left[\frac{1}{3}, \frac{2}{3}\right] \cup \left[\frac{1}{2}, \frac{3}{4}\right] \times \left\{\frac{1}{3}\right\} \cup \left\{\frac{3}{4}\right\} \times \left[\frac{1}{3}, 1\right].$$

Let $Z \subset [0, 1] \times X$ denote the pre-image of the set P under the function $\mathrm{id} \times f$:

$$Z = \{(t, x) \in [0, 1] \times X : (t, f(x)) \in P\}.$$

As P is closed and $\mathrm{id} \times f$ is continuous the set Z is compact. Define the (continuous) map $\pi : Z \to X$ by $\pi((t, x)) = x$ for every $(t, x) \in Z$.

Let $\mathscr{F}$ be the set of all clopen subsets of Z that are mapped onto X by π.

Claim 1. *The set $\mathscr{F}$ is a nonempty ultrafilter in the family of clopen subsets of Z.*

Proof. Suppose we have closed sets F and G such that $Z = F + G$. Define closed subsets A_i, B_i of X, where $i \in \{0, 1, 2\}$, by

$$A_0 = \left\{x \in X : \left(\frac{1}{4}, x\right) \in F\right\}, \quad B_0 = \left\{x \in X : \left(\frac{1}{4}, x\right) \in G\right\}$$

$$A_1 = \left\{x \in X : \left(\frac{1}{2}, x\right) \in F\right\}, \quad B_1 = \left\{x \in X : \left(\frac{1}{2}, x\right) \in G\right\}$$

$$A_2 = \left\{x \in X : \left(\frac{3}{4}, x\right) \in F\right\}, \quad B_2 = \left\{x \in X : \left(\frac{3}{4}, x\right) \in G\right\}$$

It is clear that $A_i \cap B_i = \emptyset$ for every $i \in \{0, 1, 2\}$.

If $x \in (A_0 \cap B_1) \cup (B_0 \cap A_1)$ then $f(x) < \frac{2}{3}$ as $f(x) = \frac{2}{3}$ is clearly impossible. Similarly we see that $f[(A_1 \cap B_2) \cup (B_1 \cap A_2)] \subset (\frac{1}{3}, 1]$.

Let A^* and B^* be closed sets of X, such that A^* is equal to the following union of closed sets

$$\bigcup\left\{f^{-1}\left[0, \frac{1}{3}\right] \cap A_0, f^{-1}\left[\frac{2}{3}, 1\right] \cap A_2, A_0 \cap A_1 \cap A_2, A_0 \cap B_1 \cap B_2, B_0 \cap B_1 \cap A_2, B_0 \cap A_1 \cap B_2\right\},$$

and we get a description of the closed set B^* by interchanging A's and B's in the above equation. The sets A^* and B^* are disjoint closed subsets of X and their union is the whole of X. As X is connected one of these sets must be empty. So without loss of generality we can assume that $B^* = \emptyset$. We see that $\pi[F] = X$ and furhermore, that π maps G, the complement of F into the set $f^{-1}[\frac{1}{3}, \frac{2}{3}]$, a proper subset of X.

This argument shows that if $F_1, F_2 \in \mathscr{F}$ then $\pi[X \setminus (F_1 \cap F_2)] \subset f^{-1}[\frac{1}{3}, \frac{2}{3}]$, whence $\mathscr{F}$ is seen to be a filter; it also shows that $\mathscr{F}$ is an ultrafilter. □

Let $Y \subset Z$ be given by $Y = \bigcap \mathcal{F}$ and let $\psi : Y \to X$ be the restriction of π to the continuum Y, $\psi = \pi \upharpoonright Y$.

Claim 2. *$\psi : Y \to X$ is weakly confluent.*

Proof. Suppose we have $A \subset X$ connected. If we look at the image of A under the function f there are a number of possibilities:

1. $f[A] \subset [0, \frac{2}{3}]$ and $f[A] \cap [0, \frac{1}{3}) = \emptyset$. As $\pi[Z \setminus Y] \subset f^{-1}[\frac{1}{3}, \frac{2}{3}]$ we know that $\{\frac{1}{4}\} \times A$ must intersect Y. As $\{\frac{1}{4}\} \times A$ is connected we even have that $\{\frac{1}{4}\} \times A$ is a subset of Y.
2. $f[A] \subset [\frac{1}{3}, \frac{2}{3}]$. The component Y of Z must intersect at least one of the connected subsets $\{\frac{1}{4}\} \times A$, $\{\frac{1}{2}\} \times A$ or $\{\frac{3}{4}\} \times A$ of Z, because Y is mapped onto X. And so Y must contain at least one of these connected sets.
3. $f[A] \cap [0, \frac{1}{3}) \neq f[A] \cap (\frac{2}{3}, 1]$. As above, assuming that $A^{+}(= \pi^{-1}[A]) = F + G$, we can construct closed and disjoint subsets A^* and B^* of A which cover it. Again the image under ψ is either all of A or a proper subset of A. The (unique) component of A^+ that maps onto the whole of A must intersect the set Y, and so is contained in it.

This ends the proof of the claim. □

If we let U be the set $\{(t, x) \in Y : t \in [0, \frac{3}{8}]\}$, $V = \{(t, x\} \in Y : t \in [\frac{3}{8}, \frac{5}{8}]\}$ and $W = \{(t, x) \in Y : t \in [\frac{5}{8}, 1]\}$, then $\{U, V, W\}$ is a closed cover of the space Y such that $\psi^{-1}[a] \subset U$, $\psi^{-1}[b] \subset W$, $U \cap V \cap \psi^{-1}[c] = \emptyset$, $V \cap W \cap \psi^{-1}[d] = \emptyset$ and $U \cap W = \emptyset$. This ends the proof of the lemma as it is easily seen that X and Y are both of the same weight. □

4. A topological proof of the Maćkowiak-Tymchatyn theorem

Before we start with the proof of the theorem we restate the following well known lemma on a base for the closed sets of some (transfinite) inverse sequence.

Let $\{X_\alpha, f_\alpha, \kappa\}$ be an (transfinite) inverse sequence with X_κ as its inverse limit space. Let for every $\alpha < \kappa$ the continuous function π_α be defined by $\pi_\alpha = \mathrm{proj}_\alpha \upharpoonright X_\kappa : X_\kappa \to X_\alpha$, where $\mathrm{proj}_\alpha : \Pi_{\alpha<\kappa} X_\alpha \to X_\alpha$ is the projection. The following lemma is well known.

Lemma 4. *The family of all sets of the form $\pi_\alpha^{-1}[F]$, where F is a closed subset of the space X_α and α runs over a subset C cofinal in κ, is a base for the closed sets of X_κ, the limit of the (transfinite) inverse sequence $\{X_\alpha, f_\alpha, \kappa\}$. Moreover, if for every $\alpha < \kappa$ a base $\mathfrak{B}_\alpha$ for the closed sets of space X_α is fixed, then the subfamily of those $\pi_\alpha^{-1}[F]$ for which $F \in \mathfrak{B}_\alpha$, also is a base for the closed sets of X_κ.*

Let X be a metric continuum. We are going to define a inverse sequence $\{X_n, f_n, \omega\}$,

$$X = X_0 \xleftarrow{f_0} X_1 \xleftarrow{f} \dots \xleftarrow{f_{n-1}} X_n \xleftarrow{f_n} \dots,$$

in such a way that the inverse limit space X_ω is a hereditarily indecomposable one-dimensional continuum of countable weight such that $\pi_0 : X_\omega \to X$ is a weakly confluent map onto X.

For every n we will define a metric continuum X_n, an onto map $f_n : X_n \to X_{n\;1}$ and a countable base $\mathfrak{B}_n$ for the closed sets of X_n that is closed under finite unions and intersections. Lemma 4 tells us that $\mathfrak{B} = \bigcup_{n<\omega} \pi_n^{-1}[\mathfrak{B}_n]$ will be a countable base for the closed sets of X_ω. If we choose the bases $\mathfrak{B}_n$ in such a way that $f_n^{-1}[\mathfrak{B}_{n-1}] \subset \mathfrak{B}_n$, then we even have that $\mathfrak{B}$ is closed under finite unions and intersections.

By theorem 3 and lemma 1 we know that X_ω is a one-dimensional hereditarily indecomposable continuum of countable weight if we can make sure that the base $\mathfrak{B}$ has the following two properties

1. For every $a, b, c \in \mathfrak{B}$ with empty intersection there are $A, B, C \in \mathfrak{B}$ such that $a \subset A, b \subset B, c \subset C, A \cap B \cap C = \emptyset$ and $A \cup B \cup C = X_\omega$.
2. For every $a, b, c, d \in \mathfrak{B}$ such that $a \cap b = a \cap d = b \cap c = \emptyset$ there are $U, V, W \in \mathfrak{B}$ such that $a \subset U$, $b \subset W$, $U \cap V \cap c = \emptyset$, $V \cap W \cap d = \emptyset$, $U \cap W = \emptyset$ and $X_\omega = U \cup V \cup W$.

To consider all the triples and quadruples of $\mathfrak{B}$ it is more than enough, by the definition of the bases $\mathfrak{B}_n$, to consider all the triples and quadruples of every $\mathfrak{B}_n$. As there are countably many of those we can find an enumeration σ of those triples and quadruples of length ω in such a way that the n-th element $\sigma(n)$ of this enumeration will be some triple or quadruple of some base $\mathfrak{B}_m$ with $m \leq n$.

Furthermore, if all the bonding maps f_n are weakly confluent then the map π_0 will be weakly confluent. This is easily seen: given some subcontinuum A of X we can define an inverse sequence $\{A_n, g_n, \omega\}$, where $A_0 = A$ and, for all n, A_{n+1} is some subcontinuum of X_n such that $g_n[A_{n+1}] = A_n$. Where g_n is the restriction of f_n to the set A_{n+1}. The inverse limit of this sequence is a subcontinuum of X_ω which is mapped onto A by the map π_0.

We will use lemma 2 and lemma 3 in the construction of the inverse sequence $\{X_n, f_n, \omega\}$. Suppose we have defined all X_m, f_m and $\mathfrak{B}_m$ for $m \leq n$. If $\sigma(n)$ is some triple of $\mathfrak{B}_m$ then we look at $\{a, b, c\}$, their pre-image under the map f_m^n in X_n. We use lemma 2 to find X_{n+1} and f_{n+1}, and we choose a countable base $\mathfrak{B}_{n+1}$ for the closed sets of X_n such that it contains $\{A, B, C\}$ and $f_{n+1}^{-1}[\mathfrak{B}_n]$, where $\{A, B, C\}$ is the closed cover of X_{n+1} we get from lemma 2. When $\sigma(n+1)$ was a quadruple of $\mathfrak{B}_m$ then we do something similar as above but this time we use lemma 3.

In a similar way we can construct for any continuum X, using lemmas 2 and 3 some (transfinite) inverse sequence $\{X_\alpha, f_\alpha, w(X)\}$ such that $X_0 = X$ and the inverse limit of this sequence will be a one-dimensional hereditarily indecomposable continuum of weight $w(X)$ that is mapped onto X by the weakly confluent map π_0. This provides an independent proof of a theorem in [3] which states just this. The proof in that paper made essential use of the metric case.

5. A model-theoretic proof of the Maćkowiak-Tymchatyn theorem

For the remainder of this section we fix some metric continuum X. In this section we will prove the Maćkowiak-Tymchatyn theorem for this X in two steps. First we show that X is a continuous image of some metric one-dimensional hereditarily indecomposable continuum, and then we show that the map can even be weakly confluent. Both steps will be proved by model-theoretic means, which means that that we will prove the statements by showing that some specific theory in a specific language has a model.

5.1. Preliminaries

5.1.1. Wallman spaces and lattices. In the proof we will consider the lattice of closed sets of our metric continuum X and try to find, through model-theoretic means, another lattice in which we can embed our lattice of closed sets of X. This new lattice will be a model for some sentences which will make sure that its Wallman representation is a continuum with certain properties. So at the base of the proof is Wallman's generalization, to the class of distributive lattices, of Stone's representation theorem for Boolean algebras. Wallman's representation theorem is as follows.

Theorem 4 ([6]). *If L is a distributive lattice, then there is a compact T_1 space X with a base for its closed sets that is a homomorphic image of L. If L is also disjunctive then we can find a base for its closed sets that is an isomorphic image of L.*

We call the space X the Wallman space of L or the Wallman representation of L, notation: wL.

A lattice L is *disjunctive* (or *separative*) if it models the sentence

$$\forall ab\, \exists x[a \sqcap b \neq a) \to ((a \sqcap x = x) \wedge (b \sqcap x = \mathbf{0}))]. \tag{1}$$

Furthermore the space X in theorem 4 is Hausdorff if and only if the lattice L is a normal lattice. We call a lattice normal if it models the sentence

$$\forall ab\, \exists xy[(a \sqcap b = \mathbf{0}) \to ((a \sqcap x = \mathbf{0}) \wedge (b \sqcap y = \mathbf{0}) \wedge (x \sqcup y = \mathbf{1}))]. \tag{2}$$

Note that, if we start out with a compact Hausdorff space X and look at a base for its closed subsets which is closed under finite unions and intersections, i.e., a (normal, disjunctive and distributive) lattice, then the Wallman space of this lattice is just the space X.

Remark 1. *From now on we refer to a base for the closed subsets of some topological space X, which is closed under finite unions and intersections, as a lattice base for X.*

The following theorem shows how to create an onto mapping from maps between lattices. In this theorem 2^X denotes the family of all closed subsets of the space X.

Theorem 5. [2] *Let X and Y be compact Hausdorff spaces and let $\mathscr{C}$ be a lattice base for Y. Then Y is a continuous image of X if and only if there is a map $\phi : \mathscr{C} \to 2^X$ such that*

1. $\phi(\emptyset) = \emptyset$, *and if* $F \neq \emptyset$ *then* $\phi(F) \neq \emptyset$
2. *if* $F \cup G = Y$ *then* $\phi(F) \cup \phi(G) = X$
3. *if* $F_1 \cap \ldots \cap F_n = \emptyset$ *then* $\phi(F_1) \cap \ldots \cap \phi(F_n) = \emptyset$.

So Y is certainly a continuous image of X if there is an embedding of some lattice base of the closed sets of Y into 2^X.

5.1.2. Translation of properties. Our model-theoretic proof of theorem 1 will be as follows. Given a metric continuum X, we will construct a lattice L such that some lattice base of X is embedded into L, the Wallman representation wL of L is a one-dimensional hereditarily indecomposable continuum and that for every subcontinuum in X there exists a subcontinuum of wL that is mapped onto it.

For this we need to translate things like being hereditarily indecomposable, being of dimension less than or equal to one and being connected in terms of closed sets only.

Using the characterization of hereditary indecomposability as stated in 3, we see that a compact Hausdorff space Y is hereditarily indecomposable if the lattice 2^Y models the sentence

$$\begin{aligned} (3)\qquad & \forall abcd\, \exists xyz[((a \sqcap b = \mathbf{0}) \wedge (a \sqcap d = \mathbf{0}) \wedge (b \sqcap c = \mathbf{0})) \to \\ & \to ((a \sqcap (y \sqcup z) = \mathbf{0}) \wedge (b \sqcap (x \sqcup y) = \mathbf{0}) \wedge (x \sqcap z = \mathbf{0}) \wedge \\ & \wedge (x \sqcap y \sqcap c = \mathbf{0}) \wedge (y \sqcap z \sqcap d = \mathbf{0}) \wedge (x \sqcup y \sqcup z = \mathbf{1}))]. \end{aligned}$$

Using lemma 1, we see that a space Y is of dimension less than or equal to one if the lattice 2^Y models the sentence

$$\begin{aligned} (4)\qquad & \forall abc\, \exists xyz[(a \sqcap b \sqcap c = \mathbf{0}) \to \\ & \to ((a \sqcap x = a) \wedge (b \sqcap y = b) \wedge (c \sqcap z = c) \wedge (x \sqcap y \sqcap z = \mathbf{0}) \wedge (x \sqcup y \sqcup z = \mathbf{1}))]. \end{aligned}$$

A space Y is connected if the lattice 2^Y models the sentence $\mathrm{conn}(\mathbf{1})$, where $\mathrm{conn}(a)$ is shorthand for the formula $\forall xy[((x \sqcap y = \mathbf{0}) \wedge (x \sqcup y = a)) \to (x = a) \vee (x = \mathbf{0}))]$.

5.2. The space X is a continuous image of some one-dimensional hereditarily indecomposable metric continuum

Using theorem 4 and 5 of the previous section we see that to get a hereditarily indecomposable one-dimensional continuum of countable weight that maps onto X we must find a countable distributive, disjunctive normal lattice L such that it is a model of the sentences 3, 4 and $\mathrm{conn}(\mathbf{1})$, and furthermore that some lattice base of X is embedded into this lattice L.

Fix a lattice base $\mathfrak{B}$ for X.

For some countable set of constants K we will construct a set of sentences Σ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$. We will make sure that Σ is a consistent set of sentences such that, if we have a model $\mathfrak{A} = (A, \mathscr{I})$ for Σ then

$$L(\mathfrak{A}) = \mathscr{I} \restriction K$$

is the universe of some lattice model in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\}$ which is normal, distributive and disjunctive and models the sentences 3, 4 and conn$(\mathbf{1})$. To make sure that $\mathfrak{B}$ is embedded into $L(\mathfrak{A})$ we simply add the diagram of the lattice $\mathfrak{B}$ to the set Σ and make sure that there are constants in K representing the elements of $\mathfrak{B}$. The interpretations of $\sqcap$, $\sqcup$, $\mathbf{0}$ and $\mathbf{1}$ are given by their interpretations under $\mathscr{I}$ in the model $\mathfrak{A}$.

Let K be the following countable set of counstants

$$K = \bigcup_{-1 \le n < \omega} K_n = \bigcup_{-1 \le n < \omega} \{k_{n,m} : m < \omega\}.$$

We define sets Σ_n of sentences by an ω-recursion and set $\Sigma = \bigcup_{n<\omega} \Sigma_n$.

To begin we define $K_{-1} = \mathfrak{B}$ and $\Sigma_0 = \triangle_{\mathfrak{B}}$, the diagram of $\mathfrak{B}$.

The sets Σ_n will have the following properties:

1. The Σ_{5n+1}'s will be sets of sentences that will make sure that the $L(\mathfrak{A})$ is a distributive lattice and that the Wallman space $wL(\mathfrak{A})$ of the lattice $L(\mathfrak{A})$ is connected.
2. The Σ_{5n+2}'s will be sets of sentences that will make sure that the lattice $L(\mathfrak{A})$ is normal.
3. The Σ_{5n+3}'s will be sets of sentences that will make sure that $L(\mathfrak{A})$ is a disjunctive lattice.
4. The Σ_{5n+4}'s will be sets of sentences that will make sure that the lattice $L(\mathfrak{A})$ will be a model of the sentence 4.
5. And the $\Sigma_{5(n+1)}$'s will be sets of sentences that will make sure that the lattice $L(\mathfrak{A})$ will be a model of the sentence 3.

5.2.1. Construction of Σ in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$. We now how to define the sets of sentences of $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \bigcup_{m<5n+4} K_m$ as described in $1-5$.

We have a natural order $\lhd$ on the set $K = \bigcup_m K_m$ defined by

$$k_{n,m} \lhd k_{r,t} \leftrightarrow [(n < r) \vee ((n = r) \wedge (m < t))].$$

Let $\{p_l\}_{l<\omega}$ be an enumeration of

$$\Big\{p \in \Big[\bigcup_{m \le 5n} K_m\Big]^2 : p \setminus \bigcup_{m \le 5(n-1)} K_m \neq 0\Big\}.$$

For every $l < \omega$ write $p_l = \{p_l(0), p_l(1)\}$.

$$\Sigma^0_{5n+1} = \{p_l(0) \sqcap p_l(1) = k_{5n+1,2l} : l < \omega\}$$
$$\Sigma^1_{5n+1} = \{p_l(0) \sqcup p_l(1) = k_{5n+1,2l+1} : l < \omega\}$$

Furthermore we let Σ^2_{5n+1} be a set of sentences in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \bigcup_{m\le 5n} K_m$ (without quantifiers) consisting of

1. sentences that state that according to the constants from $\bigcup_{m\le 5n} K_m$ we are dealing with a distributive lattice with a $\mathbf{0}$ and a $\mathbf{1}$,
2. sentences that make sure that no pair of constants from will refute conn($\mathbf{1}$).

Define Σ_{5n+1} by

$$\Sigma_{5n+1} = \Sigma^0_{5n+1} \cup \Sigma^1_{5n+1} \cup \Sigma^2_{5n+1}.$$

This set of sentences will make sure that any model of Σ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$ will be a distributive lattice with a $\mathbf{0}$ and a $\mathbf{1}$, and also a model of the sentence conn($\mathbf{1}$).

$$\Sigma_{5n+2} = \{[(p_l(0) \sqcap p_l(1) = \mathbf{0}) \to ((p_l(1) \sqcap k_{5n+2,2l} = \mathbf{0}) \wedge \\ \wedge (p_l(0) \sqcap k_{5n+2,2l+1} = \mathbf{0}) \wedge (k_{5n+2,2l} \sqcup k_{5n+2,2l+1} = \mathbf{1}))] : l < \omega\}.$$

This set of sentences will make sure that any (lattice) model of Σ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$ will be normal.

The following set of sentences makes sure that any model of Σ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$ which is also a lattice is a disjunctive lattice.

$$\Sigma^0_{5n+3} = \{[(p_l(0) \sqcap p_l(1) \neq p_l(0)) \to ((k_{5n+3,2l+1} \sqcap p_l(0) = k_{5n+3,2l+1}) \wedge \\ \wedge (k_{5n+3,2l+1} \sqcap p_l(1) = \mathbf{0}))] : l < \omega\}$$

$$\Sigma^0_{5n+3} = \{[(p_l(1) \sqcap p_l(0) \neq p_l(1)) \to ((k_{5n+3,2l} \sqcap p_l(1) = k_{5n+3,2l}) \wedge \\ \wedge (k_{5n+3,2l} \sqcap p_l(0) = \mathbf{0}))] : l < \omega\}.$$

And define Σ_{5n+3} by

$$\Sigma_{5n+3} = \Sigma^0_{5n+3} \cup \Sigma^1_{5n+3}.$$

Let ζ denote the following formula in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\}$

$$\zeta(a, b, c; x, y, z) = [(a \sqcap b \sqcap c = \mathbf{0}) \to ((a \sqcap x = a) \wedge (b \sqcap y = b) \wedge (c \sqcap z = c) \wedge \\ \wedge (x \sqcap y \sqcap z = \mathbf{0}) \wedge (x \sqcup y \sqcup z = \mathbf{1}))].$$

Let $\{q_l\}_{l<\omega}$ be an enumeration of the set

$$\Big\{q \in \Big[\bigcup_{m\le 5n} K_m\Big]^3 : q \setminus \bigcup_{m\le 5(n-1)} K_m \neq \emptyset\Big\}.$$

For every $l < \omega$ writte $q_l = \{q_l(0), q_l(1), q_l(2)\}$.

Now define Σ_{5n+4} by

$$\Sigma_{5n+4} = \{\zeta(q_l(0), q_l(1), q_l(2); k_{5n+4,3l}, k_{5n+4,3l+1}, k_{5n+4,3l+2}) : l < \omega\}.$$

This will make sure that the Wallman space of any lattice model of Σ will be at most one-dimensional.

For making sure that the Wallman space of any model of Σ will be hereditarily indecomposable we introduce the following formulas in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\}$:

$$\phi(a, b, c, d) = [(a \sqcap b = \mathbf{0}) \wedge (a \sqcap d = \mathbf{0}) \wedge (b \sqcap c = \mathbf{0})]$$
$$\begin{aligned}\psi(a, b, c, d; x, y, z) = [&(x \sqcup y \sqcup z = \mathbf{1}) \wedge (x \sqcap z = \mathbf{0}) \wedge \\ &\wedge (a \sqcap (y \sqcup z) = \mathbf{0}) \wedge (b \sqcap (x \sqcup y) = \mathbf{0}) \wedge \\ &\wedge (x \sqcap y \sqcap c = \mathbf{0}) \wedge (y \sqcap z \sqcap d = \mathbf{0})]\end{aligned}$$
$$(5) \qquad \theta(a, b, c, d; x, y, z) = \phi(a, b, c, d) \rightarrow \psi(a, b, c, d; x, y, z)$$

Let $\{r_l\}_{l<\omega}$ be an enumeration of the set

$$\left\{r \in {}^4\left[\bigcup_{m \leq 5n} K_m\right] : \operatorname{ran}(r) \setminus \bigcup_{m \leq 5(n-1)} K_m \neq \emptyset\right\}.$$

Let $\Sigma_{5(n+1)}$ be the set of sentences defined by:

$$\Sigma_{5(n+1)} = \{\theta(r_l(0), r_l(1), r_l(2), r_l(3); k_{5(n+1),3l}, k_{5(n+1),3l+1}, k_{5(n+1),3l+2}) : l < \omega\}.$$

Here the formula θ is as in equation 5.

5.2.2. Consistency of Σ in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$. In this section we show that Σ is a consistent set of sentences by finding, for every finite subset Γ of Σ a metric space Y and an interpretation function $\mathscr{I} : K \rightarrow 2^Y$ such that $(2^Y, \mathscr{I})$ is a model for the theory $\Gamma \cup \triangle_{\mathfrak{B}}$. The interpretations of $\sqcap$, $\sqcup$, $\mathbf{0}$ and $\mathbf{1}$ will always be $\cap$, $\cup$ (the normal set intersection and union), $\emptyset$ and Y respectively.

For $\Gamma = \emptyset$ we let $Y = X$ and we interpret every constant from K_{-1} as its corresponding base element in $\mathfrak{B}$. Extend the interpretation function by assigning the empty set to all constants of $K \setminus K_{-1}$. It is obvious that $(2^Y, \mathscr{I})$ is a model of $\triangle_{\mathfrak{B}}$.

Remark 2. *As the interpretation of $\sqcap$ and $\sqcup$ in the metric continuum Y will always be the normal set intersection and set union, all the sentences in Σ^2_{5n+1} for some $n < \omega$ are true in the model $(2^Y, \mathscr{I})$. So we can ignore these sentences and for the remainder of this section concentrate on the remaining sentences of Σ.*

We can define a well order $\sqsubset$ on the set $\Sigma \setminus \{\Sigma^2_{5n+1} : n < \omega\}$ by stating that $\phi \sqsubset \psi$ if and only if there are $n < m < \omega$ such that $\phi \in \Sigma_n$ and $\psi \in \Sigma_m$ or there are $k < l < \omega$ and $n < \omega$ such that $\phi, \psi \in \Sigma_n$ and ϕ is a sentence that mentions p_k (q_k or r_k respectively) and ψ is a sentence that mentions p_l (q_l or r_l respectively).

Suppose Γ is a finite subset of Σ such that each of its proper subsets has a model as stated as above. Let θ be the $\sqsubset$-largest sentence in $\Gamma \setminus \{\Sigma^2_{5n+1} : n < \omega\}$ and let Y be a metric continuum and $\mathscr{I} : K \rightarrow 2^Y$ be an interpretation function such that $(2^Y, \mathscr{I})$ is a model of the theory $\Gamma \setminus \{\theta\} \cup \triangle_{\mathfrak{B}}$.

We will show that there exists a metric space Z and an interpretation function $\mathscr{J} : K \rightarrow 2^Z$ such that $(2^Z, \mathscr{J})$ is a model of the theory $\Gamma \cup \triangle_{\mathfrak{B}}$. We consider three cases: $\theta \in \Sigma_{5n+1} \cup \Sigma_{5n+2} \cup \Sigma_{5n+3}$, $\theta \in \Sigma_{5n+4}$ and $\theta \in \Sigma_{5(n+1)}$ for some $n < \omega$.

1. If $\theta \in \Sigma_{5n+1} \cup \Sigma_{5n+2} \cup \Sigma_{5n+3}$, we can simply let $Z = Y$ and either interpret the new constant under $\mathscr{J}$ as the intersection or union of two closed sets in Y if θ is in some Σ_{5n+1} or, if θ is an element of some Σ_{5n+2} or Σ_{5n+3}, using

the fact that the space Y is normal find $\mathscr{I}$-interpretations for the newest constants, in an obvious way.

2. If $\theta \in \Sigma_{5n+4}$, then θ is a sentence of the following form

$$\theta = \left[(a \sqcap b \sqcap c = \mathbf{0}) \rightarrow (a \sqcap x = a) \wedge (b \sqcap y = b) \wedge \wedge (c \sqcap z = c) \wedge (x \sqcap y \sqcap z = \mathbf{0}) \wedge (x \sqcup y \sqcup z = \mathbf{1})\right].$$

Suppose the preamble of θ is true in the model $(2^Y, \mathscr{I})$. If a has a zero interpretation then we can choose $x = \mathbf{0}$, $y = \mathbf{1}$ and $z = \mathbf{1}$, and this interpretation of x, y and z makes sure that θ holds in the model $(2^Y, \mathscr{I})$. So we may assume that a, b and c have non zero interpretations.

By lemma 2 there exist a metric continuum Z, a closed, monotone and onto map $f: Z \rightarrow Y$, and a closed cover $\{A, B, C\}$ of Z with empty intersection such that $f^{-1}[\mathscr{I}(a)] \subset A$, $f^{-1}[\mathscr{I}(b)] \subset B$ and $f^{-1}[\mathscr{I}(c)] \subset C$.

Define an interpretation function $\mathscr{J}: K \rightarrow 2^Z$ by

$$\mathscr{J}(k) = f^{-1}[\mathscr{I}(k)] \text{ for all } k \in K \setminus \{x, y, z\}$$
$$\mathscr{J}(x) = A, \ \mathscr{J}(y) = B \text{ and } \mathscr{J}(z) = C.$$

With this interpretation function $(2^Z, \mathscr{J})$ is a model for Γ.

3. If $\theta \in \Sigma_{5(n+1)}$ then it is of the form $\theta(a, b, c; x, y, z)$ as in equation 5. Suppose the preamble of θ is true in the model $(2^Y, \mathscr{I})$.

If the interpretation of a is zero we can simply take $x = y = \mathbf{0}$ and $z = \mathbf{1}$ to make $(2^Y, \mathscr{I})$ a model of θ. So we may again assume that the interpretations of a, b, c and d are nonzero.

By lemma 3 there exists a metric continuum Z, a weakly confluent onto map $f: Z \rightarrow Y$ and a closed cover $\{U, V, W\}$ of Z such that $f^{-1}[\mathscr{I}(a)] \subset V$, $f^{-1}[\mathscr{I}(b)] \subset W$, $U \cap V \cap f^{-1}[\mathscr{I}(c)] = \emptyset$, $U \cap W = \emptyset$, and $V \cap W \cap f^{-1}[\mathscr{I}(d)] = \emptyset$.

Define an interpretation function $\mathscr{J}: K \rightarrow 2^Z$ by

$$\mathscr{J}(k) = f^{-1}[\mathscr{I}(k)] \text{ for all } k \in K \setminus \{x, y, z\}$$
$$\mathscr{J}(x) = U, \ \mathscr{J}(y) = V \text{ and } \mathscr{J}(z) = W.$$

The structure $(2^Z, \mathscr{J})$ is a model for Γ.

So the theory Σ is a consistent theory in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$.

5.3. The Maćkowiak-Tymchatyn theorem

Apart from the weakly confluent property of the continuous onto map we have proven the Maćkowiak-Tymchatyn theorem, theorem 1.

In this section we will extend the language of the previous section and construct a consistent theory in this extended language that shows that there exists a one-dimensional hereditarily indecomposable continuum Y (of the form $wL(\mathfrak{A})$) that maps onto the continuum X by a weakly confluent map. By this approach the weight of the continuum Y will be greater than the weight of the our space X. We can amend this by taking a countable elementary sublattice of $L(\mathfrak{A})$.

To make sure that the continuous map following from the previous section is weakly confluent, we must consider all the subcontinua of the space X.

We let $\hat{\mathcal{K}}$ be the following set

$$\hat{\mathcal{K}} = \bigcup_{-2 \le n < \omega} \hat{\mathcal{K}}_n = \bigcup_{-2 \le n < \omega} \{k_{n,\alpha} : \alpha < |2^X|\}.$$

We will construct a theory $\hat{\Sigma} = \bigcup_{-1 \le n < \omega} \hat{\Sigma}_n$ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{\mathcal{K}}$ similar as in the previous section such that given any model $\mathfrak{A} = (A, \mathscr{I})$ of $\hat{\Sigma}$, the set $L(\mathfrak{A}) = \mathscr{I} \restriction \hat{\mathcal{K}}$ will be the universe of some normal distributive and disjunctive lattice such that it is a model of the sentences 3, 4 and conn$(\mathbf{1})$, we can embed the lattice 2^X into $L(\mathfrak{A})$, so there exists a continuous map f from $wL(\mathfrak{A})$ onto X and, for every subcontinuum of X there exists a subcontinuum of $wL(\mathfrak{A})$ that is mapped onto it by f.

5.3.1. Construction of $\hat{\Sigma}$ in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{\mathcal{K}}$. We let $\hat{\mathcal{K}}_{-1} = \{k_{-1,\alpha} < |2^X|\}$ correspond to the set $2^X = \{x_\alpha : \alpha < |2^X|\}$ in such a way that the set of all the subcontinua of X corresponds to the set $\{x_{2_\alpha} : \alpha < \beta\}$ for some ordinal number $\beta \le |2^X|$. Let the set of sentences $\hat{\Sigma}_0$ in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{\mathcal{K}}_{-1}$ correspond to $\triangle_{2^X}$, the diagram of the lattice 2^X.

We want to define a set of sentences $\hat{\Sigma}_{-1}$ in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{\mathcal{K}}_{-2} \cup \hat{\mathcal{K}}_{\ 1}$ that will make sure that if $\mathfrak{A}$ is a model of $\hat{\Sigma}$ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{\mathcal{K}}$ then we have for every subcontinuum in X a subcontinuum of $wL(\mathfrak{A})$ that will be mapped onto it by the continuous onto map we get by the fact that 2^X is embedded in the lattice $L(\mathfrak{A})$.

$$\begin{aligned}
\hat{\Sigma}^0_{-1} &= \{\mathrm{conn}(k_{-2,\alpha}) \wedge (k_{-2,\alpha} \sqcap k_{-1,\alpha}) : \alpha < \beta\} \\
\hat{\Sigma}^1_{-1} &= \{(\mathrm{conn}(k_{-2,\alpha}) \wedge (k_{-2,\alpha} \sqcap k_{-1,\gamma} = k_{-2,\alpha})) \to \\
&\qquad \to (k_{-1,\alpha} \sqcap k_{-1,\gamma} = k_{-1,\alpha}) : \alpha < \beta, \gamma < |2^X|\} \\
\hat{\Sigma}^2_{-1} &= \{k_{-2,\gamma} = \mathbf{0} : \beta \le \gamma < |2^X|\}.
\end{aligned}$$

And define the set of sentences $\hat{\Sigma}_{-1}$ as $\hat{\Sigma}_{-1} = \hat{\Sigma}^0_{-1} \cup \hat{\Sigma}^1_{-1} \cup \hat{\Sigma}^2_{-1}$.

Suppose $\mathfrak{A}$ is a model of $\hat{\Sigma}$. The set $\hat{\Sigma}^0_{-1}$ will make sure that for every subcontinuum C of X there is some subcontinuum C' of $wL(\mathfrak{A})$ that is mapped into C by the continuous onto map f we get from theorem 5 and the fact that 2^X is embedded into $wL(\mathfrak{A})$. The set $\hat{\Sigma}^1_{-1}$ will then make sure that C' is in fact mapped onto C by the map f.

Let us further construct the sets $\hat{\Sigma}_n$ for $0 < n < \omega$ in the same manner as we have constructed the set Σ_n in the previous section. So that if we have a model $\mathfrak{A}$ of $\hat{\Sigma}$, the lattice $L(\mathfrak{A})$ will be a normal distributive and disjunctive lattice that models the sentences 3, 4 and conn$(\mathbf{1})$.

5.3.2. Consistency of $\hat{\Sigma}$ in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{\mathcal{K}}$. Suppose we go about as in section 5.2.2 and try to prove by that given a model $(2^Y, \mathscr{I})$ for the theory Γ and γ a sentence of $\hat{\Sigma}$ constructed after the sentences from Γ, that there exist a model $(2^Z, \mathscr{I})$ for the theory $\Gamma \cup \{\gamma\}$, either by using lemma 2 or 3 or the fact that Y and Z

are metric continua. A problem may arise if we use lemma 3 to find the space Z, as in this case $f: Z \to Y$ is only weakly confluent so we cannot just take the f-inverse image of the $\mathscr{I}$-interpretation of constants from $\hat{K}_{-2}$ as their $\mathscr{J}$-interpretations, as these might not be connected. We can however always find a connected subset that maps onto the $\mathscr{I}$-interpretation under the map f. These $\mathscr{J}$-interpretations of the c_i's (might) affect all other $\mathscr{J}$-interpretations, and it could happen that some sentence in Γ true in the model $(2^Y, \mathscr{I})$, because its premise was false, has now a true premise in $(2^Z, \mathscr{J})$ and we have to find $\mathscr{J}$-interpretations for the constants introduced by this sentence to make it a true sentence in $(2^Z, \mathscr{J})$. This again could affect the interpretations and the truth value of other sentences in Γ, and so on.

To bypass this problem we will consider every finite set Γ of $\hat{\Sigma}$ separately, and find a model for it.

We fix such a finite set Γ from now on.

Note that there is only mention of finitely many constants $\{c_1, \ldots, c_k\}$ from the set $\hat{K}_{-2}$. We start by construction a model $(2^{X^+}, \mathscr{I}^+)$ from X which is not only a model of $\triangle_{2^X}$ and all the sentences from $\Gamma \cap \hat{\Sigma}_{-1}$ but also models $c_i \sqcap c_j = \mathbf{0}$ for all $i \neq j$. Denote $\Gamma \setminus \hat{\Sigma}_{-1}$ by $\{\gamma_1, \ldots, \gamma_m\}$ in such a way that the γ in $\Gamma \setminus \hat{\Sigma}_{-1}$ which are in $\hat{\Sigma}_m$ have lower index than those in $\hat{\Sigma}_n$, when $m < n$. We will construct models $(2^Y, \mathscr{I}_i)$ such that

$$(2^Y, \mathscr{I}_i) \models (\Gamma \cap \hat{\Sigma}_{-1}) \cup \triangle_{2^X} \cup \{c_i \sqcap c_j = \mathbf{0} : i \neq j\} \cup \{\gamma_1, \ldots, \gamma_i\}.$$

All these metric continua are related in the following way

$$X^+ \xleftarrow{g_1} Y_1 \xleftarrow{g_2} Y_2 \xleftarrow{g_3} \ldots \xleftarrow{g_i} Y_i,$$

where the g_i's are either the identity map or come from lemma 2 or 3.

Construction of $(2^{X^+}, \mathscr{I}^+)$. Note that the constants from $\hat{K}_{-2} \cup \hat{K}_{-1}$ correspond with closed sets from the metric continuum X. So c_i corresponds with some subcontinuum C_i of X, and $a \in \hat{K}_{-1}$ corresponds to some closed set A of X.

Let X^+ be the space $X \times [0, 1]$ and define the interpretation map $\mathscr{I}^+ : \hat{K} \to 2^{X^+}$ by

$$\mathscr{I}^+(a) = \pi_X^{-1}[A] \quad \text{for all } a \in \hat{K}_{-1}$$

$$\mathscr{I}^+(c_i) = C_i \times \left\{\frac{i}{k}\right\} \quad \text{for all } i$$

$$\mathscr{I}^+(a) = \emptyset \quad \text{for all } a \in \hat{K} \setminus (\hat{K}_{-1} \cup \{c_1, \ldots, c_k\})$$

By construction, we have

$$(2^{X^+}, \mathscr{I}^+) \models (\Gamma \cap \hat{\Sigma}_{-1}) \cup \triangle_{2^X} \cup \{c_i \sqcap c_j = \mathbf{0} : i \neq j\}.$$

Suppose now that we have already taken care of the sentences $\{\gamma_1, \ldots, \gamma_{i-1}\}$ of Γ. We will show how to find a model $(2^Y, \mathscr{I}_i)$ for different γ_i. Let $Y = Y_{i-1}$ and $\mathscr{I} = \mathscr{I}_{i-1}$. We have

$$(2^Y, \mathscr{I}^+) \models (\Gamma \cap \hat{\Sigma}_{-1}) \cup \triangle_{2^X} \cup \{c_i \sqcap c_j = \mathbf{0} : i \neq j\} \cup \{\gamma_j : j < i\}.$$

The sentence γ_i is in $\hat{\Sigma}_{5n+j}$ for some $j = 1, 2, 3, 4$. If γ_i is some sentence in one of the sets $\hat{\Sigma}_{5n+1}, \hat{\Sigma}_{5n+2}$ or $\hat{\Sigma}_{5n+3}$ for some n then we let $Z = Y$, let the $\mathscr{J}$-interpretation of all constants of $\hat{K}_{-1}$ and those that are mentioned in some γ_i with $j < i$ equal their $\mathscr{I}$-interpretation. We use normal set intersection or union to find interpretations for the constants introduced by γ_i if it is an element of some $\hat{\Sigma}_{5n+1}$ if necessary, and normality of the space Y to find interpretations for the constants introduced by γ_i if it is a sentence in $\hat{\Sigma}_{5n+2}$ or $\hat{\Sigma}_{5n+3}$ for some n.

If γ_i is a sentence in one of the sets $\hat{\Sigma}_{5n+4}$ and its premise is true in the model $(2^Y, \mathscr{I})$ and there is no triple in 2^Y that can make the sentence a true sentence in 2^Y, then we use lemma 2 to find a continuum Z and a closed monotone map $f : Z \to Y$ such that if we let the $\mathscr{J}$-interpretation of $a \in \hat{K}$, a not equal to one of the constants introduced by γ_i be defined by

$$\mathscr{J}(a) = f^{-1}[\mathscr{I}(a)],$$

then with the interpretation of the constants introduced by γ_i by the closed sets of Z we get from the lemma, we made, with this interpretation $\mathscr{J}$ the sentence γ_i a true sentence in $(2^Z, \mathscr{J})$. None of the other sentences is affected by this construction as we take pre-images of their $\mathscr{I}$-interpretations as their $\mathscr{J}$-interpretations, and by the fact that f is closed and monotone.

The sentence γ_i is in $\hat{\Sigma}_{5(n+1)}$. Without loss of generality we can assume that the premise of $\gamma_i = \gamma_i(a, b, c; x, y, z)$, but not its conclusion is true in the model $(2^Y, \mathscr{I})$. With the aid of lemma 3 we find a metric continuum Z and closed sets U, V and W of Z such that

$$(6) \qquad \gamma_i[f^{-1}[\mathscr{I}(a)], f^{-1}[\mathscr{I}(b)], f^{-1}[\mathscr{I}(c)]; U, V, W]$$

is a true statement. We will show how to find an interpretation map $\mathscr{J} : \hat{K} \to 2^Z$ such that $(2^Z, \mathscr{J})$ models the sentences from $\{\gamma_j : j \leq i\}$, $\triangle_{2^X}$ and $\{c_i \sqcap c_j : i \neq j\}$.

1. Choose $\mathscr{J}(c_i) \subset f^{-1}[\mathscr{I}(c_i)]$ such that it is a continuum that is mapped onto $\mathscr{I}(c_i)$ by the map f.
2. Let the $\mathscr{J}$-interpretation of all the constants from $\hat{K}_{-1}$ be equal to the f-inverse of their $\mathscr{I}$-interpretation.

With the interpretation map $\mathscr{J}$ we have so far we already have that $(2^Z, \mathscr{J})$ is a model of the theory $(\Gamma \cap \hat{\Sigma}_{-1}) \cup \triangle_{2^X} \cup \{c_i \sqcap c_j = \mathbf{0} : i \neq j\}$.

Now we will consider the sentences from $\Gamma \backslash \hat{\Sigma}_{-1} = \{\gamma_j : j < i\}$ one at a time in the order given by their index. These γ_j's will be restrictions on the $\mathscr{J}$-interpretation of constants for which we have so far no $\mathscr{J}$-interpretation in 2^Z. We will find $\mathscr{J}$-interpretation for a constant a introduced by one of the γ_j's inside the f-pre-image of the $\mathscr{I}$-interpretation of a. So far all constants mentioned in some γ_j with $j < i$ we have

$$\mathscr{J}(a) \subset f^{-1}[\mathscr{I}(a)].$$

Note that so far we have that $f[\mathscr{J}(a)] = \mathscr{I}(a)$ for all constants a which $\mathscr{J}$-interpretation we have determined. We will make sure that when we consider the next γ_j in the list and find a $\mathscr{J}$-interpretation of the introduced constant a it has the following two properties

1. If $x \in \mathscr{I}(a) \cap \mathscr{I}(c_i)$ and $y \in \mathscr{J}(c_i)$ is such that $f(y) = x$, we have $y \in \mathscr{J}(a)$.
2. If for all i we have $x \notin \mathscr{I}(c_i)$ and $x \in \mathscr{I}(a)$ then we have $f^{-1}(x) \subset \mathscr{J}(a)$.

This will make sure that the premise of the next γ_j to consider (if it has any) has the same truth value in the model $(2^Y, \mathscr{I})$ as it has in the model $(2^Z, \mathscr{J})$ we have constructed so far, as for any finite number of constants $a_1, \ldots, a_n$ for which we have defined its $\mathscr{J}$-interpretation we have

$$f[\mathscr{J}(a_1) \cap \ldots \cap \mathscr{J}(a_n)] = \mathscr{I}(a_1) \cap \ldots \cap \mathscr{I}(a_n).$$

1. Suppose γ_j is of the form $a = b \sqcap c$ or $a = b \sqcup c$. The $\mathscr{J}$-interpretation of a is fully prescribed by $\mathscr{J}(b)$ and $\mathscr{J}(c)$, and it easily seen that $\mathscr{J}(a)$ has the properties 1 and 2 above if $\mathscr{J}(b)$ and $\mathscr{J}(c)$ have it.
2. Suppose γ_j is of the form $b \not\leq c \rightarrow a \leq b \wedge a \sqcap c = \mathbf{0}$. If the premise is false then $\mathscr{I}(a) = \emptyset$ and thus $\mathscr{J}(a) = \emptyset$ will suffice.

   If the premise is true $\mathscr{I}(a)$ is a nonempty closed set that witnesses that $\mathscr{I}(b)$ is not a subset of $\mathscr{I}(c)$. We choose the $\mathscr{I}$-interpretation of a by

   $$\mathscr{J}(a) = f^{-1}[\mathscr{I}(a)] \cap \mathscr{J}(b).$$

   As $\mathscr{J}(b)$ maps onto $\mathscr{I}(b)$ under the map f and as $\mathscr{I}(a)$ is a nonempty subset of $\mathscr{I}(b)$ we see that with this $\mathscr{J}$-interpretation of a, we have a witness for $b \not\leq c$ in $(2^Z, \mathscr{J})$. It is also easily seen that $\mathscr{J}(a)$ has properties 1 and 2 if $\mathscr{J}(b)$ has these properties.
3. Suppose a is one of the constants introduced by γ_j from some $\hat{\Sigma}_{5n+i}$ where $i = 2, 4, 5$. The $\mathscr{J}$-interpretation of these constants will be given by

   $$\mathscr{J}(a) = f^{-1}[\mathscr{I}(a)].$$

   This will make the sentence we are considering a true sentence in $(2^Z, \mathscr{J})$. Again $\mathscr{J}(a)$ will have properties 1 and 2.

   The closed subsets U, V and W of Z we got from lemma 3 will make the sentence γ_i a true sentence in the model $(2^Z, \mathscr{J})$ as the premise of this sentence has the same truth value as in the model $(2^Y, \mathscr{I})$ and the $\mathscr{J}$-interpretation of the contants mentioned in the premise are subsets of the f-inverse of their $\mathscr{I}$-interpretation.

All the constants for which we have not yet determined a $\mathscr{J}$-interpretation will have the f-inverse image of their $\mathscr{I}$-interpretation as their $\mathscr{J}$-interpretation, which of course is the empty set.

Remark 3. *This consistency proof also shows that there will be a set of disjoint continua in the Wallman representation of the lattice $L(\mathfrak{A})$, where $\mathfrak{A}$ is a model of $\hat{\Sigma}$ that will map onto all the continua in X by the map given by theorem 5.*

5.3.3. The Maćkowiak-Tymchatyn theorem. As $\hat{\Sigma}$ is a consistent theory in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{K}$ there is some model $\mathfrak{A}$ for it. This model gives us a normal distributive and disjunctive lattice $L(\mathfrak{A})$ which models the sentences 3, 4 and conn($\mathbf{1}$). There also exists, using the interpretations of the constants in $\hat{K}_{-1}$, an embedding of 2^X into the $L(\mathfrak{A})$. So the Wallman space $wL(\mathfrak{A})$, is a one-dimensional hereditarily indecomposable continuum which admits a weakly confluent map onto the metric continuum X.

Now we only have to make sure that there exists such a space that is of countable weight to complete the proof of the Maćkowiak-Tymchatyn theorem.

Theorem 6. [3] *Let $f : Y \to X$ be a continuous surjection between compact Hausdorff spaces. Then f can be factored as $h \circ g$, where $Y \xrightarrow{g} Z \xrightarrow{h} X$ and Z has the same weight as X and shares many properties with Y (for instance, if Y is one-dimensional so is X or if Y is hereditarily indecomposable, so is X).*

Proof. Let $\mathfrak{B}$ a minimal sized lattice-base for the closed sets of X, and identify it with its copy $\{f^{-1}[B] : B \in \mathfrak{B}\}$ in 2^Y. By the Löwenheim-Skolem theorem there is an elementary sublattice of 2^Y, of the same cardinality as $\mathfrak{B}$ such that $\mathfrak{B} \subset D \prec 2^Y$. The space wD is as required. □

Applying this theorem to the space $wL(\mathfrak{A})$ and the weakly confluent map $f : wL(\mathfrak{A}) \to X$ we get a one-dimensional hereditarily indecomposable continuum wD which admits a weakly confluent map onto the space X and moreover the weight of the space wD equals the weight of the space X. This is exactly what we were looking for.

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