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On D.C. Mappings and Differences of Convex Operators

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Introduction

Let C be an open convex set in a (real) normed linear space X. A real-valued function f on C is *d.c.* if it can be represented as the difference of two continuous convex functions on C. (For a survey about d.c. functions see [3].)

In this article we study relationships between two possible generalizations of the notion of a d.c. function to mappings between normed spaces: "d.c. mapping" and "order d.c. mappings".

Let $(Y, \preccurlyeq)$ be an ordered normed space. A mapping $G : C \to Y$ is a *convex operator* if $G((1-t)x + ty) \preccurlyeq (1-t)G(x) + tG(y)$ whenever $x, y \in C$ and $0 \leq t \leq 1$.

Definition 1. Let X, Y be normed linear spaces, $C \subset X$ be an open convex set, and $F : C \to Y$ be a mapping.

(a) F is a *d.c. mapping* on C if there exists a continuous convex function $f : C \to \mathbb{R}$ (*control function*) such that $y^* \circ F + f$ is a continuous convex function on C for each $y^* \in Y^*$ with $\|y^*\| \leq 1$.

(b) If $(Y, \preccurlyeq)$ is an ordered normed space, F is *order d.c.* if F can be represented as the difference of two continuous convex operators on C.

The notion of a d.c. mapping was introduced by the authors and widely studied in [5], where a theory of d.c. mappings was built. In contrast to order d.c. mappings, the class of d.c mappings is quite stable.

It is easy to see that the notions from Definition 1 are equivalent for $Y = \mathbb{R}^n$ (equipped with the standard coordinate-wise partial ordering). The situation is much more complicated for infinite-dimensional Y.

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The articles [5] and [1] contain examples of order d.c. mappings which are not d.c. Namely, Proposition 21 of [1] says that for each separable normed linear space X there exists a mapping $F : X \to \ell_2$ which is order d.c. but is d.c. on no open convex set $C \subset X$. (Note that it is not difficult to prove by the same method that the same holds with an arbitrary normed linear space X and ℓ_p $(1 < p \leq \infty)$, instead of ℓ_2.)

In the present paper, we consider the implication "d.c. implies order d.c." for mappings $F : \mathbb{R}^d \to Y$, where $Y = \ell_p$, $Y = c_0$ or Y is a member of a large general class of sequence spaces.

The main consequences of our results are the following.

a) Each d.c. mapping $F : \mathbb{R} \to \ell_p$ $(1 \leq p \leq \infty)$ is order d.c.

b) There exists a d.c. mapping $F : \mathbb{R} \to c_0$ which is order d.c. on no open interval.

c) For each $1 \leq p \leq \infty$ there exist an integer $d \geq 2$ and a d.c. mapping $F : \mathbb{R}^d \to \ell_p$ which is order d.c. on no open convex set.

Note that the case $Y = \ell_\infty$ is exceptional and almost trivial—each d.c. mapping $F : X \to \ell_\infty$ (where X is an arbitrary normed linear space) is order d.c. Indeed, if $f : X \to \mathbb{R}$ is a control function for F, then $G := (f, f, \ldots)$ and $G - F$ are clearly continuous convex operators and therefore $F = G - (G - F)$ is order d.c.

In the sequel we will need the following characterization of d.c. mappings of one real variable.

Theorem 2 ([5]). *Let $I \subset \mathbb{R}$ be an open interval, Y be a normed linear space. Given a mapping $F : I \to Y$, the following are equivalent:*

(i) *F is d.c. on I;*

(ii) *the right derivative $F'_+(x)$ exists for each $x \in I$ and F'_+ has locally finite variation on I.*

We shall use the following notations for balls: B_X is the closed unit ball of a normed linear space X, $B(a, r)$ denotes the open r-ball centered in a.

Results

We are going to show that d.c. mappings of one real variable are order d.c. for a large class of sequence spaces, requiring the following definition. (Similar spaces were considered in [4].)

Definition 3. Let Γ be a nonempty set, and $\|\cdot\| : \mathbb{R}^\Gamma \to [0, \infty]$ be a *norm*, i.e. a function which is convex, even, positively homogeneous and attains the value 0 only at the origin. We denote by $S_{\|\cdot\|}(\Gamma)$ the ordered normed space

$$S_{\|\cdot\|}(\Gamma) = \{y \in \mathbb{R}^\Gamma : \|y\| < \infty\}$$

with the norm $\|\cdot\|$ and the standard pointwise partial ordering.

Theorem 4. *Let $I \subset \mathbb{R}$ be an open interval, Γ be a nonempty set. Let a norm $\|\cdot\| : \mathbb{R}^\Gamma \to [0, \infty]$ have the following properties:*

(a) $\|y\| < \infty$ *whenever y has a finite support;*

(b) $\|y\| \leq K \cdot \sup\{\|y\chi_{\Gamma_0}\| : \Gamma_0 \subset \Gamma$ *is finite*$\}$ *for some $K > 0$ and each $y \in \mathbb{R}^\Gamma$;*

(c) $\|y\| \leq \|z\|$ *whenever $y, z \in \mathbb{R}^\Gamma$, $|y| \leq |z|$.*

Then each d.c. mapping $F : I \to S_{\|\cdot\|}(\Gamma)$ is order d.c.

Proof. For $\gamma \in \Gamma$ and $x \in I$ denote $F_\gamma(x) = F(x)(\gamma)$. It easily follows from (c) that each projection $y \mapsto y(\gamma)$ is continuous. Using this fact and Theorem 2, it is easy to see that $g_\gamma(x) := F'_+(x)(\gamma) = (F_\gamma)'_+(x)$. Fix $x_0 \in I$ and put

$$f_\gamma(x) = \int_{x_0}^{x} V_{x_0}^t g_\gamma \, dt\,,$$

where $V_a^b \varphi$ denotes the variation of a function φ on the interval $[a, b]$, with the obvious change of the sign if $a > b$. By [5] (Theorem 2.3, proof of the implication (*iii*) $\Rightarrow$ (*i*)), for each γ, F_γ is controlled by f_γ.

Define $f : I \to \mathbb{R}^\Gamma$ by $f(x)(\gamma) = f_\gamma(x)$. It remains to show that the values of f belong to $S_{\|\cdot\|}(\Gamma)$ and f is continuous as a mapping into $S_{\|\cdot\|}(\Gamma)$. (Indeed, then f will be a continuous convex operator such that also $F + f$ is a continuous convex operator).

Consider a finite set $\Gamma_0 \subset \Gamma$ and $\varepsilon > 0$. Let $x \in I$, $x > x_0$. There exists a partition $\{x_0 = s_0 < s_1 < \ldots < s_n = x\}$ such that, for each $\gamma \in \Gamma_0$,

$$V_{x_0}^x g_\gamma \leq \varepsilon + \sum_{i=1}^{n} |g_\gamma(s_i) - g_\gamma(s_{i-1})|\,.$$

Let $e_\gamma \in \mathbb{R}^\Gamma$ be the characteristic function of the singleton $\{\gamma\}(\gamma \in \Gamma)$. Then we have (using (c))

$$\begin{aligned}
\|f(x)\chi_{\Gamma_0}\| &= \left\| \sum_{\gamma \in \Gamma_0} \left(\int_{x_0}^{x} V_{x_0}^t g_\gamma \, dt \right) e_\gamma \right\| \leq (x - x_0) \left\| \sum_{\gamma \in \Gamma_0} (V_{x_0}^x g_\gamma)\, e_\gamma \right\| \\
&\leq (x - x_0)\,\varepsilon \left\| \sum_{\gamma \in \Gamma_0} e_\gamma \right\| + (x - x_0) \left\| \sum_{i=1}^{n} \sum_{\gamma \in \Gamma_0} |g_\gamma(s_i) - g_\gamma(s_{i-1})|\, e_\gamma \right\| \\
&= (x - x_0)\,\varepsilon \|\chi_{\Gamma_0}\| + (x - x_0) \left\| \sum_{i=1}^{n} |F'_+(s_i) - F'_+(s_{i-1})|\, \chi_{\Gamma_0} \right\| \\
&\leq (x - x_0)\,\varepsilon \|\chi_{\Gamma_0}\| + (x - x_0) \sum_{i=1}^{n} \|F'_+(s_i) - F'_+(s_{i-1})\, \chi_{\Gamma_0}\| \\
&\leq (x - x_0)\,\varepsilon \|\chi_{\Gamma_0}\| + (x - x_0) \sum_{i=1}^{n} \|F'_+(s_i) - F'_+(s_{i-1})\| \\
&\leq (x - x_0)\,\varepsilon \|\chi_{\Gamma_0}\| + (x - x_0)\, V_{x_0}^x F'_+\,.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have proved that $\|f(x)\chi_{\Gamma_0}\| \leq (x - x_0)\, V_{x_0}^x F'_+$. Then (*b*) implies that $f(x) \in S_{\|\cdot\|}(\Gamma)$. For $x < x_0$ the proof is analogous.

Let us prove that f is $\|\cdot\|$-continuous. Let $[\alpha, \beta] \subset I$ be an arbitrary interval containing x_0. For $\alpha \leq x_1 < x_2 \leq \beta$ and a finite set $\Gamma_0 \subset \Gamma$, we have

$$\|(f(x_2) - f(x_1)\chi_{\Gamma_0}\| = \left\| \sum_{\gamma \in \Gamma_0} \int_{x_1}^{x_2} V_{x_0}^t g_\gamma \, \mathrm{d}t \cdot e_\gamma \right\| \leq (x_2 - x_1) \left\| \sum_{\gamma \in \Gamma_0} V_\alpha^\beta g_\gamma \cdot e_\gamma \right\|.$$

Proceeding as above we obtain that $\|(f(x_2) - f(x_1))\chi_{\Gamma_0}\| \leq (x_2 - x_1) V_\alpha^\beta F'_+$. Since $V_\alpha^\beta F'_+$ is finite, (b) implies that f is Lipschitz on $[\alpha, \beta]$. □

Corollary 5. *Let Γ be a nonempty set, $I \subset \mathbb{R}$ be an open interval, $1 \leq p \leq \infty$. Then every d.c. mapping $F : I \to \ell_p(\Gamma)$ is order d.c.*

The following example shows that the assertion of Theorem 4 fails if the range space is c_0. A mapping is called *nowhere order d.c.* if it is order d.c. on no open convex set.

Proposition 6. *There exists a d.c. mapping $G : \mathbb{R} \to c_0$ which is nowhere order d.c.*

Proof. For each $n \in \mathbb{N}$ define $g_n : \mathbb{R} \to \mathbb{R}$ by: $g_n(2k/n) = 0, g_n((2k+1)/n) = 1\, n$ $(k \in \mathbb{Z})$ and g_n is affine on each interval $\left[\frac{j}{n}, \frac{j+1}{n}\right]$. Then g_n is Lipschitz with constant $1, 0 \leq g_n(x) \leq 1/n$ for each $x \in \mathbb{R}$, and $V_\alpha^\beta g_n = \beta - \alpha$ for each interval $[\alpha, \beta] \subset \mathbb{R}$. It follows that $g(x) := (g_1(x), g_2(x), \dots)$ defines a 1-Lipschitz mapping of $\mathbb{R}$ into c_0. Then the mapping $G(x) := \int_0^x g(t)\, \mathrm{d}t$ is d.c. on $\mathbb{R}$ (cf. Theorem 2).

Suppose that G is order d.c. on some open interval $I \subset \mathbb{R}$. There exists a continuous convex operator $F = (F_1, F_2, \dots) : I \to c_0$ such that the two mappings $\pm G + F$ are continuous convex operators on I. (Indeed, if $G = G_1 - G_2$, where G_1, G_2 are continuous convex operators, we can put $F := G_1 + G_2$.) Denoting $f_n := (F_n)'_+ \ (n \in \mathbb{N})$, it follows that all the real functions $\pm g_n + f_n$ are non-decreasing on I. This easily implies that $f_n(\beta) - f_n(\alpha) \geq V_\alpha^\beta g_n = \beta - \alpha$ whenever $\alpha < \beta$ are points from I.

Consider three points $a < b < c$ from the interval I. By convexity,

$$\frac{F_n(b) - F_n(a)}{b - a} \leq f_n(b) \leq \frac{F_n(c) - F_n(b)}{c - b}$$

which implies that $f_n(b) \to 0$ as $n \to \infty$. Since $f_n(x) \geq f_n(b) + (x - b)$ for each $x \in [b, c]$, we get

$$F_n(c) - F_n(b) = \int_b^c f_n(x)\, \mathrm{d}x \geq (c - b) f_n(b) + \frac{(c - b)^2}{2}.$$

But this contradicts the fact that $\lim F_n(c) = \lim F_n(b) = \lim f_n(b) = 0$. □

Now we are going to construct an example (Proposition 9) showing that the assertion of Corollary 5 does not hold for arbitrary finite-dimensional domain space instead of $\mathbb{R}$. In what follows, λ_d denotes the d-dimensional Lebesgue measure in $\mathbb{R}^d$, and $\lambda := \lambda_1$.

Lemma 7. *Let positive numbers R, r, c such that $R/4 > r$ be given. Let f, g, h be real functions on $(-R, R)$ such that $f = g - h$, g and h are convex, $f(-r) = f(r) = 0$ and $f(0) = c$. Then*

$$\lambda\left\{x : \frac{R}{4} \leq |x| \leq R, |h(x)| \geq \frac{cR}{8r}\right\} \geq \frac{R}{4}. \tag{1}$$

Proof. Convexity of g implies $g(r) + g(-r) - 2g(0) \geq 0$. Since $f(r) + f(-r) - 2f(0) = -2c$ and $h = g - f$, we obtain $h(r) + h(-r) - 2h(0) \geq 2c$. Elementary properties of convex functions imply that

$$h'_+(r) \geq h'_-(r) \geq r^{-1}(h(r) - h(0)) \quad \text{and} \quad h'_-(-r) \leq h'_+(-r) \leq r^{-1}(h(0) - h(-r)).$$

Consequently

$$h'_+(r) = h'_-(-r) \geq r^{-1}(h(r) + h(-r) - 2h(0)) \geq \frac{2c}{r}. \tag{2}$$

To prove (1), it is sufficient to prove that at least one of the intervals $I_1 := (-R, -3R/4)$, $I_2 := (-R/2, -R/4)$, $I_3 := (R/4, R/2)$, $I_4 := (3R/4, R)$ is a subset of $\{x : |h(x)| \geq (8r)^{-1} cR\}$. Suppose to the contrary that there exist points $x_1 \in I_1$, $x_2 \in I_2$, $x_3 \in I_3$, $x_4 \in I_4$ such that $|h(x_i)| < (8r)^{-1} cR$, $i = 1, 2, 3, 4$. Then we clearly have

$$h'_+(r) \leq h'_+(x_3) \leq \frac{h(x_4) - h(x_3)}{x_4 - x_3} < \frac{2(8r)^{-1} cR}{R/4} = \frac{c}{r},$$

$$h'_-(-r) \geq h'_-(x_2) \geq \frac{h(x_2) - h(x_1)}{x_2 - x_1} > \frac{-2(8r)^{-1} cR}{R/4} = -\frac{c}{r}$$

and $h'_+(r) - h'_-(-r) < 2c/r$ which contradicts to (2). □

We will need also the following easy lemma.

Lemma 8. *Let $d \geq 2$ be a natural number and $1 \leq p < d$ be a real number. Then in the d-dimensional open unit ball $B(0, 1) \subset \mathbb{R}^d$ there exists a sequence $B(x_n, r_n)$, $n = 1, 2, \dots$ of pairwise disjoint open balls such that $x_n \to 0$ and $\sum_{n=1}^{\infty} (r_n)^p = \infty$.*

Proof. First observe that in an arbitrary open ball U there exists a finite system of pairwise disjoint open balls $\mathscr{F}(U) = \{B(y_1, \rho_1), \dots, B(y_k, \rho_k)\}$ such that $\sum_{i=1}^{k} (\rho_i)^p \geq 1$. To this end denote for each $\varepsilon > 0$ by $V(U, \varepsilon)$ the maximal number of disjoint open balls which have radius ε and are subsets of U. It is easy to see that $\varepsilon^{\ d} = O(V(U, \varepsilon))$, $\varepsilon \to 0+$. Since $\varepsilon^{-d}\varepsilon^p \to \infty$, $\varepsilon \to 0+$, existence of $\mathscr{F}(U)$ easily follows. Now choose a sequence $U_n = B(z_n, a_n)$, $n = 1, 2, \dots$ of disjoint open balls such that $z_n \to 0$. Order all members of the system $\bigcup_{n=1}^{\infty} \mathscr{F}(U_n)$ in an arbitrary way to a sequence $B(x_n, r_n)$. It is easy to see that it has the desired properties. □

Proposition 9. *Let $d \geq 2$ be a natural number and let p be a real number such that $1 \leq p < d$. Then there exists a d.c. mapping $F : \mathbb{R}^d \to \ell_p$ which is order d.c. on no open convex neighbourhood of $0 \in \mathbb{R}^d$. Moreover, F is bounded and controlled by a function $K\|\cdot\|^2$, where $K > 0$ and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$.*

Proof. By Lemma 8 we can find in $B(0,1) \subset \mathbb{R}^d$ a sequence $B(x_n, r_n)$, $n = 1, 2, \ldots$ of pairwise disjoint open balls such that $x_n \to 0$ and $\sum_{n=1}^{\infty} (r_n)^p = \infty$. It is well-known that there exists a C^∞ function φ on $\mathbb{R}^d$ such that supp $\varphi \subset B(0, 1/2)$ and $\varphi(0) = 1$. Put

$$f_n(x) := (r_n)^2 \varphi\left(\frac{x - x_n}{r_n}\right), \qquad x \in \mathbb{R}^d$$

and $F(x) := (f_1(x), f_2(x), \ldots)$. Clearly F is a mapping to ℓ_p.

It is easy to see that the derivative φ' is Lipschitz on $\mathbb{R}^d$ with a constant $K > 0$. Therefore Proposition 1.11 of [5] gives that φ is d.c. on $\mathbb{R}^d$ with the control function $q(x) := K\|x\|^2$, where $\|\cdot\|$ is the Eucliden norm. It is easy to see (cf. Lemma 1.5. of [5]) that $f_n(x) = (r_n)^2 \varphi((r_n)^{-1}(x - x_n))$ is d.c. on $\mathbb{R}^d$ with the control function $q_n(x) = (r_n)^2 q((r_n)^{-1}(x - x_n)) = K\|x - x_n\|^2$. Since the function $q_n - q$ is clearly affine, we see that q is a control function of each f_n. (This can be deduced also from the fact that each f_n has a K-Lipschitz derivative.)

Now we are ready to prove that F is a d.c. mapping. To this end consider the mappings $F_n(x) := (f_1(x), f_2(x), \ldots, f_n(x), 0, 0, \ldots\}$. Clearly all F_n are continuous mappings $\mathbb{R}^d \to \ell_p$. Fix arbitrary

$$y^* \in (l_p)^* = \ell^q, \quad y^* = (y_1, y_2, \ldots), \quad \|y^*\|_q = 1 \text{ and } n \in \mathbb{N}.$$

Since $|y_i| \leq 1$, $i = 1, \ldots, n$, and the sets $\overline{\text{supp } f_i}$, $i = 1, \ldots, n$ are pairwise disjoint, we easily see that the function

$$y^* \circ F_n + q = \sum_{i=1}^{n} y_i f_i + q$$

is locally convex and therefore also convex on $\mathbb{R}^d$. Therefore each F_n is a d.c. mapping with the control function q. Since $F_n(x) \to F(x)$, $x \in \mathbb{R}^d$, and both F and q are clearly bounded on a ball, Corollary 1.15 of [5] implies that also F is d.c. on $\mathbb{R}^d$ with the control function q.

Now suppose to the contrary that F is an order d.c. mapping on a neighbourhood U of 0 and choose $R > 0$ such that $B(0, 3R) \subset U$. By definition, there exist continuous convex operators

$$G, H : B(0, 3R) \to \ell_p, \quad G = (g_1, g_2, \ldots), \quad H = (h_1, h_2, \ldots)$$

such that $F = G - H$ on $B(0, 3R)$. Thus all g_i and h_i are convex real functions on $B(0, 3R)$ and $f_n = g_n - h_n$ on $B(0, 3R)$ for each $n \in \mathbb{N}$.

Because $x_n \to 0$ and also $r_n \to 0$ (since $\sum (r_n)^d$ clearly converges), we can find an index $n_0 \in \mathbb{N}$ such that $B(x_n, r_n) \subset B(0, R)$ and $r_n < R/4$ for each $n \geq n_0$. Now fix an index $n \geq n_0$ and a vector $u \in \mathbb{R}^d$, $\|u\| = 1$, and consider real functions

$$f(t) := f_n(x_n + tu), \quad g(t) := g_n(x_n + tu), \quad h(t) := h_n(x_n + tu)$$

for $t \in (-R, R)$. Applying Lemma 7 with $r := r_n$ and $c := (r_n)^2$, we obtain

$$\lambda\{t : R/4 < |t| < R, |h_n(x_n + tu)| > (R/8)\, r_n\} \geq R/4. \tag{3}$$

Let $S := \{u \in \mathbb{R}^d : \|u\| = 1\}$ and denote by ν the surface measure on S. Applying to $h^*(z) := |h_n(x_n + z)|^p$ the well-known formula (cf. [2], 3.2.13) rewritten using Fubini theorem, we obtain

$$I := \int_{B\,0,R)} h^*(z)\, \mathrm{d}\lambda_d(z) = \int_S \left(\int_0^R r^{d-1} h^*(ru)\, \mathrm{d}r \right) \mathrm{d}\nu(u).$$

Using the fact that ν is invariant w.r.t. the mapping $u \mapsto -u$ $(u \in S)$, and (3), we easily obtain

$$I = (1/2) \int_S \left(\int_{-R}^R |r|^{d-1} h^*(ru)\, \mathrm{d}r \right) \mathrm{d}\nu(u) \geq (1/2)\, \nu(S)\, (R/4)^d\, (Rr_n/8)^p$$

Consequently, since $B(0, 2R) \supset B(x_n, R)$, we have

$$\int_{B(0,2R)} |h_n(z)|^p\, \mathrm{d}\lambda_d(z) \geq \int_{B(x_n, R)} |h_n(z)|^p\, \mathrm{d}\lambda_d(z) = I \geq C(r_n)^p,$$

where $C > 0$ is a constant which does not depend on n. Thus

$$\int_{B(0,2R)} \sum_{n=n_0}^{\infty} |h_n(z)|^p\, \mathrm{d}\lambda_d(z) = \sum_{n=n_0}^{\infty} \int_{B(0\ 2R)} |h_n(z)|^p\, \mathrm{d}\lambda_d(z) \geq C \sum_{n=n_0}^{\infty} (r_n)^p = \infty.$$

On the other hand, the real function $(\|H(x)\|_p)^p$ is continuous on $\overline{B(0, 2R)} \subset B(0, 3R)$ and therefore

$$\int_{B(0,2R)} \sum_{n=n_0}^{\infty} |h_n(z)|^p\, \mathrm{d}\lambda_d(z) \leq \int_{B(0,2R)} (\|H(x)\|_p)^p\, \mathrm{d}\lambda_d(z) < \infty.$$

which is a contradiction.

Since φ and $\{r_n\}$ are bounded and the functions f_n have disjoint supports, F is bounded. □

Using Proposition 9, it is possible to accumulate singularities to obtain d.c. mapping that are nowhere order d c.

Proposition 10. *Let $d \in \mathbb{N}$ and $p \in \mathbb{R}$ be such that $d \geq 2$ and $1 \leq p < d$. Then there exists a d.c. mapping $F : \mathbb{R}^d \to \ell_p$ which is nowhere order d.c.*

Proof. By Proposition 9, there exists a bounded d.c. mapping $G : \mathbb{R}^d \to \ell_p$, $G = (G_1, G_2, \ldots)$, controlled by $\|\cdot\|^2$, such that G is order d.c. on no open convex neighbourhood of 0. (Indeed, it is sufficient to put $G := \frac{1}{K}F$.)

Let $M > 0$ be such that $\|G(x)\| \leq M$ for each $x \in \mathbb{R}^d$.

Fix a sequence $\{x_n\}$ which is dense in $\mathbb{R}^d$ and positive numbers c_n such that

$$c_n \cdot \max\{1, \|x_n\|, \|x_n\|^2\} \leq 2^{-n}.$$

For $x \in \mathbb{R}^d$, define

$$F_{n,k}(x) = c_n G_k(x - x_n) \qquad ((n, k) \in \mathbb{N} \times \mathbb{N})$$

$$f(x) = \sum_{n=1}^{\infty} c_n \|x - x_n\|^2.$$

The choice of $\{c_n\}$ easily implies that the (convex) function f is bounded on bounded sets and therefore continuous. The functions $F_{n,k}$ define a bounded mapping

$$F : \mathbb{R}^d \to \ell_p(\mathbb{N} \times \mathbb{N}), \qquad F := (F_{n,k})_{(n,k) \in \mathbb{N} \times \mathbb{N}}$$

since

$$\|F(x)\|_p^p = \sum_n c_n^p \sum_k |G_k(x - x_n)|^p = \sum_n c_n^p \|G(x - x_n)\|_p^p \leq M^p \sum_n 2^{-np}.$$

Let $y^* = (y^*_{n,k}) \in \ell_q(\mathbb{N} \times \mathbb{N}) = \ell_p(\mathbb{N} \times \mathbb{N})^*$ be such that $\|y^*\|_q \leq 1$. For each n, consider the element $y^*_n = (y^*_{n,k})_{k=1}^{\infty}$ of $\ell_q = (\ell_p)^*$, and the function

$$\varphi_n(x) := \sum_{k=1}^{\infty} y^*_{n,k} \cdot G_k(x - x_n) + \|x - x_n\|^2 = y^*_n \circ G(x - x_n) + \|x - x_n|^2$$

which is convex and continuous, since $\|y^*_n\|_q \leq 1$ and G is controlled by $\|\cdot\|^2$. Then also the function

$$y^* \circ F + f = \textstyle\sum_n c_n \varphi_n$$

is convex. Moreover, it is also continuous being bounded on bounded sets (indeed, F is bounded and f is bounded on bounded sets). Thus F is a d.c. mapping controlled by the function f.

Let us show that F is order d.c. on no open convex set $C \subset \mathbb{R}^d$. Choose $n_0 \in \mathbb{N}$ such that $x_{n_0} \in C$, and consider the following continuous linear projection

$$P : \ell_p(\mathbb{N} \times \mathbb{N}) \to \ell_p, \qquad (y_{n,k})_{(n,k)} \mapsto (y_{n_0,k})_k.$$

Then P is order-preserving and $P \circ F(x) = c_{n_0} G(x - x_{n_0})$. Thus $P \circ F$ is not order d.c. on C. Consequently, neither F is order d.c. on C.

Since $\ell_p(\mathbb{N} \times \mathbb{N})$ can be identified with ℓ_p, we have proved that there exists a d.c. mapping $G : \mathbb{R}^d \to \ell_p$ which is nowhere order d.c. □

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