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Parametric Extension of the Poincaré Theorem

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The structure of the zeros set $f^{-1}(\mathbf{0})$ of a continuous function $f: I^{n+1} \to R^n$, $I = [0, 1]$, satisfying some additional boundary conditions are investigated. This gives an extension of some classical results due to Bolzano, Poincaré, Brouwer, Eilenberg and Otto.

§1. A main result. Let $I^n := [0, 1]^n$ be the n-dimensional cube of the Euclidean space R^n and let us denote by

$$I_i^- := \{x \in I^n : x(i) = 0\}, \qquad I_i^+ := \{x \in I^n : x(i) = 1\}$$

its i-th opposite faces. In this paper we are going to prove the following.

Theorem. *Let $\{(H_i^-, H_i^+) : i = 1, \ldots, n\}$ be a family of pairs of closed sets such that $I_i^- \times I \subset H_i^-$, $I_i^+ \times I \subset H_i^+$ and $I^n \times I = H_i^- \cup H_i^+$.*

Then there exists a connected set $W \subset \bigcap_{i=1}^n H_i^- \cap H_i^+$ such that

$$W \cap (I^n \times \{0\}) \neq \emptyset \neq W \cap (I^n \times \{1\}).$$

The proof of this theorem will be based on two combinatorial lemmas.

Letting $H_i^- := f_i^{-1}(-\infty, 0]$, $H_i^+ := f_i^{-1}[0, \infty)$ we obtain a parametric extension of Poincaré's theorem (cf. [7, 3, 4]):

Corollary 1. *Let $f: I^n \times I \to R^n$, $f = (f_1, \ldots, f_n)$, $I = [0, 1]$, be a continuous map such that for each $i \leq n$*

$$f_i(I_i^- \times I) \subset (-\infty, 0] \text{ and } f_i(I_i^+ \times I) \subset [0, \infty).$$

Then there exists a connected set $W \subset f^{-1}(\mathbf{0})$ such that

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$$W \cap (I^n \times \{0\}) \neq \emptyset \neq W \cap (I^n \times \{1\}).$$

It is easy to observe that Corollary 1 implies Theorem. It suffices to consider functions $f_i(x) := d(x, H_i^-) - d(x, H_i^+)$, $i = 1, \ldots, n$, where $d(x, A) := \inf\{\|x - a\| : a \in A\}$ means the distance functions from a set A.

From Corollary 1 we immediately obtain an extension of Brouwer's theorem due to Browder (cf. [1, 6]):

Corollary 2. *If $g : I^n \times I \to I^n$ is a continuous map then there is a connected set $W \subset \{(x, t) \in I^n \times I : g(x, t) = x\}$ such that*

$$W \cap (I^n \times \{0\}) \neq \emptyset \neq W \cap (I^n \times \{1\}).$$

Indeed, the map $f(x, t) := x - g(x, t)$ satisfies the assumptions of Corollary 1.

A closed subset F of a topological space X is a *partition* between two sets $A_0, A_1 \subset X$ if there are two disjoint open sets $U_0, U_1 \subset X$ such that $X \backslash F = U_0 \cup U_1$ and $A_i \subset U_i$ for $i = 0, 1$.

The following corollary is an extension of the Eilenberg-Otto theorem [2]:

Corollary 3. *Let $F_1, \ldots, F_n$ be closed subsets of the cube $I^n \times I$ such that each set F_i's is a partition between $I_i^- \times I$ and $I_i^+ \times I$. Then the intersection $F_1 \cap \ldots \cap F_n$ contains a connected set $W \subset F_1 \cap \ldots F_n$ such that*

$$W \cap (I^n \times \{0\}) \neq \emptyset \neq W \cap (I^n \times \{1\}).$$

Proof. Let $U_i^\varepsilon \subset I^n \times I$, $i = 1, \ldots, n$, $\varepsilon = -, +$ be open sets such that $I^n \times I \backslash F_i = U_i^- \cup U_i^+$, $U_i^- \cap U^+ = \emptyset$, $I_i^- \times I \subset U^-$, $I_i^+ \times I \subset U_i^+$. The sets $H_i^- := U_i^- \cup F_i$ and $H_i^+ := U_i^+ \cup F_i$ satisfy the assumptions of Theorem. $\square$

§2. A combinatorial part. Let $k > 1$ be a given natural number and let $Z_k := \{i/k : i \in Z\}$, where Z denotes the set of integers. Let Z_k^n denote the Cartesian product of n copies of the set Z_k:

$$Z_k^n := \{z : \{1, \ldots, n\} \to Z_k \mid z \text{ is a map}\}.$$

Using the Cartesian notation let $\mathbf{0} := (0, \ldots, 0)$ be the neutral element and let $e_i := (0, \ldots, 0, 1/k, 0, \ldots, 0)$, $e_i(i) = 1/k$, be the i-th basic vector. Denote by $P(n)$ the set of permutations of the set $\{1, \ldots, n\}$.

An ordered set $S = [z_0, \ldots, z_n] \subset Z_k^n$ is said to be an *n-simplex* if there exists a permutation $\alpha \in P(n)$ such that

$$z_1 = z_0 + e_{\alpha(1)}, \qquad z_2 = z_1 + e_{\alpha(2)}, \ldots, \qquad z_n = z_{n-1} + e_{\alpha(n)}.$$

Any subset $[z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n] \subset S$, $i = 0, \ldots, n$, is said to be the $(n-1)$-*face* of the n-simplex S. A subset $C \subset Z_k^n$ of the form

$$C := C(k) = \left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\right\}^n$$

is said to be a *combinatorial n-cube.* For $n > 1$ let us define the *i-th combinatorial back and front faces* of C as

$$C_i^- := C_i^-(k) = \{z \in C : z(i) = 0\}, \qquad C_i^+ := C_i^+(k) = \{z \in C : z(i) = 1\},$$

and the *boundary* as

$$\partial C := \bigcup \{C_i^- \cup C_i^+ : i = 1, \ldots, n\}.$$

In the case $n = 1$ let us put $C = \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}$ and $C_1^- = \{0\}, C_1^+ = \{1\}$. The set $C = \{0\}$ is said to be *0-cube* (and *0-simplex*, too).

Let us say that an $(n-1)$-face σ of an n-simplex S *lies* in the boundary ∂C if $\sigma \subset C_i^\varepsilon$ for some $i = 1, \ldots, n$ and $\varepsilon = -, +$.

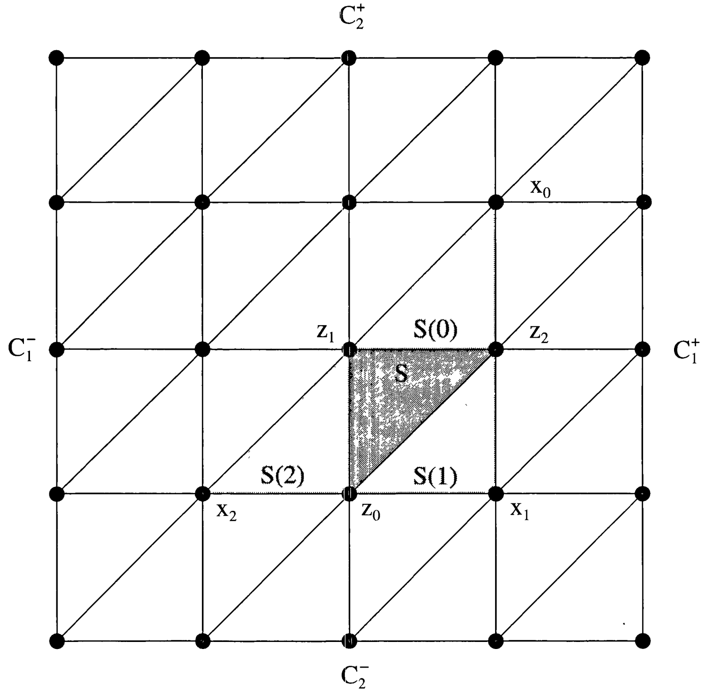

Figure 1

Observation 1. *Let $S = [z_0, \ldots, z_n] \subset Z_k^n$ be an n-simplex. Then for each point $z_i \in S$ there exists exactly one n-simplex $S[i]$ such that*

$$S \cap S[i] = \{z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n\}.$$

Proof. We shall define the *i-neighbour* $S[i]$ of the simplex S (see Figure 1) as

(a) If $0 < i < n$, then $S[i] := [z_0, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n]$, where $x_i = z_{i-1} + (z_{i+1} - z_i) = z_{i-1} + e_{\alpha(i+1)}$.

(b) If $i = 0$, then $S[0] := [z_1, \ldots, z_n, x_0]$, where $x_0 = z_n + (z_1 - z_0)$.

(c) If $i = n$, then $S[n] := [x_n, z_0, \ldots, z_{n-1}]$, where $x_n = z_0 + (z_{n-1} - z_n)$.

We leave it to the reader to prove that the n-simplexes $S[i]$ are well-defined and that they are the only possible i-neighbours of the n-simplex S. □

The following observation is immediate:

Observation 2. *Any $(n-1)$-face of an n-simplex contained in the combinatorial n-cube C is an $(n-1)$-face of exactly one or two n-simplexes from C, depending on whether or not it lies on the boundary ∂C.*

For a given map $\phi : C \to \{0, \ldots, n\}$ a subset $S \subset C$ is said to be *k-colored* if $\phi(S) = \{0, \ldots, k\}$.

First Combinatorial Lemma. *Let $\phi : C \to \{0, \ldots, n\}$ be a map an n-cube $C = C(k)$ which for $n \geq 1$ satisfies the boundary condition*

$$(\alpha) \qquad i \notin \phi(C_i^-) \quad \text{and} \quad i - 1 \notin \phi(C_i^+).$$

Then the number ρ of the all n-colored n-simplices is odd.

Proof. Before starting the proof let us note that in the case $n = 1$ the condition (α) means that $\phi(0) = 0$ and $\phi(1) = 1$. The condition (α) implies also that the face C_n^- is the only C_i^ε face which is $(n-1)$-colored. It is clear that the lemma is true for $n = 0$ because $\phi(C) = \{0\}$.

We shall proceed to the proof with the induction on n. Assume that the lemma holds for an $(n-1)$-cube, $n \geq 1$. According to the assumption (α) any $(n-1)$-colored face σ of an n-simplex which lies in ∂C lies in C_n^-. Considering C_n^- to be an $(n-1)$-cube, by our inductive hypothesis the number η of such faces is odd. Let $\eta(S)$ denotes the number of $(n-1)$-colored faces of an n-simplex $S \subset C$.

If S is an n-colored n-simplex, clearly $\eta(S) = 1$; while if S is not n-colored, we have $\eta(S) = 2$ or $\eta(S) = 0$ according as S is $(n-1)$-colored or $\{0, \ldots, n-1\} \setminus \phi(S) \neq \emptyset$. Hence

$$\rho = \sum \eta(S), \text{ mod } 2.$$

On the other hand, an $(n-1)$-colored face is counted exactly once or twice in $\sum \eta(S)$ according as it is in the boundary ∂C or not.

Accordingly

$$\sum \eta(S) = \eta, \textit{ mod } 2,$$

hence

$$\eta = \rho, \text{ mod } 2.$$

But η is odd. Thus ρ is odd, too. □

Consider the product $D := C \times J$ of a combinatorial n-cube $C = C(k)$ and an 1-cube $J = J(k) = \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}$. The set D is a combinatorial $(n+1)$-cube. Fix a map $\phi : D \to \{0, \ldots, n\}$. In the set of the all $(n+1)$-simplices contained in D let us establish the relation $\sim$; $S_1 \sim S_2$ whenever $\phi(S_1 \cap S_2) = \{0, \ldots, n\}$, i.e. $S_1 \cap S_2$ is n-colored.

From the pigeon hole principle it follows that each $(n+1)$-simplex $S \subset D$ which is n-colored has one or two simplices $S_1, S_2 \subset D$ such that $S_1 \sim S$ and $S_2 \sim S$ depending on whether S has or not n-colored face lying in ∂C.

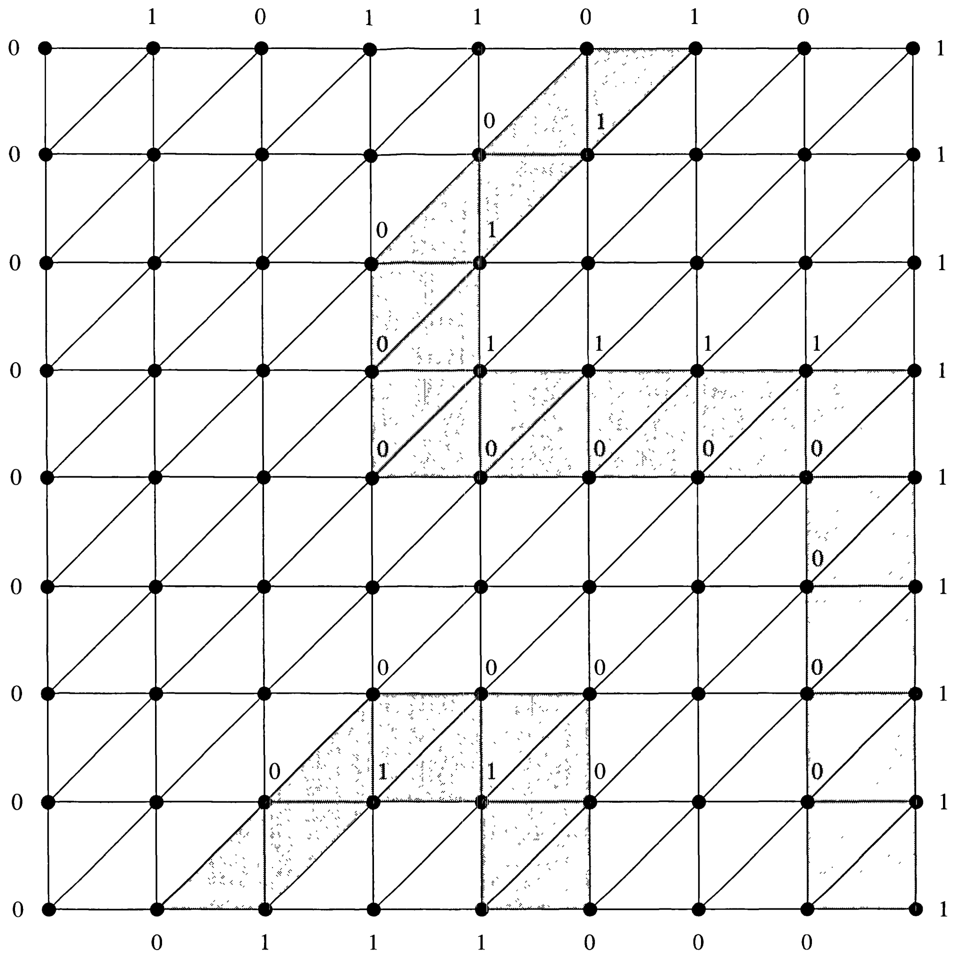

Figure 2

Second Combinatorial Lemma. *Let $\phi : C \times J \to \{0, \ldots, n\}$ be a map from the product of a combinatorial n-cube $C = C(k)$ and a combinatorial 1-cube $J = J(k)$. Assume that for each $i = 1, \ldots, n$ the following condition holds:*

$$(\beta) \qquad i \notin \phi(C_i^- \times J) \quad \text{and} \quad i - 1 \notin \phi(C_i^+ \times J).$$

Then the number of the all chains $S_0 \sim \dots \sim S_m$ of $(n+1)$-simplices such that

$$\phi(S_0 \cap (C \times \{0\})) = \{0, \dots, n\} = \phi(S_m \cap (C \times \{1\}))$$

is odd.

Proof. Consider maximal chains $S_0 \sim \dots \sim S_m$ of $(n+1)$-simplices in $C \times J$ such that
(1) $\phi(S_0 \cap (C \times \{0\})) = \{0, \dots, n\}$.
According to the boundary condition (β) there are only two possibilities (see Figure 2, $n = 2$);
(2) $\phi(S_m \cap (C \times \{0\})) = \{0, \dots, n\}$
or
(3) $\phi(S_0 \cap (C \times \{1\})) = \{0, \dots, n\}$.

From First Combinatorial Lemma it follows that the number ρ of the all $(n+1)$-simplices $S \subset C \times J$ such that $S \cap (C \times \{0\})$ is n-colored, is odd. Since any maximal chain which satisfies the conditions (1) and (2) occupies two $(n+1)$-simplices having n-colored faces in $C \times \{0\}$, so we infer that there is an odd number of chains such that the conditions (1) and (3) holds (see Figure 2 for $n = 2$). □

§ 3. A topological part. For a given sequence $\{A_n : n \in N\}$ of subsets of a metric space X let us define the set Ls $\{A_n : n \in N\}$; $x \in$ Ls $\{A_n : n \in N\}$ if and only if there exists an infinite set $M \subset N$ of points $x_m \in A_m$ such that $x = \lim \{x_m : m \in M\}$.

Lemma (see [5; Th. 5.47.6]). *Let $\{A_m : m \in N\}$ be a sequence of connected subsets of a compact metric space X such that some sequence $\{a_n : n \in N\}$ of points $a_n \in A_n$ is converging in X. Then the set $A :=$ Ls $\{A_n : n \in N\}$ is compact and connected.*

Proof. 1. First, let us prove that A is a closed. Fix $x \in X \setminus A$. Then there exists a neighbourhood U_x of x such that U_x meets only finite number of the sets A_n's. It is clear that $U_x \cap A = \emptyset$. Thus the set $X \setminus A$ is open.

2. Let $\{a_n : n \in N\}$ be a sequence of points $a_n \in A_n$ converging to a point $a \in X$. Suppose that there are two disjoint nonempty open sets $U_0, U_1 \subset X$ such that $A \subset U_0 \cup U_1$ and $U_0 \cap U_1 = \emptyset$. Assume that $a \in U_0$ and fix a point $x \in U_1 \cap A$. Let $\{x_m : m \in M\}$, be a sequence such that $x = \lim \{x_m : m \in M\}$. Observe, that for some $m \in M$; $A_m \subset U_0 \cup U_1$. Because if not then we can choose a converging subsequence $\{y_s : s \in S\}$, $S \subset M$, such that $y_s \in A_s \setminus (U_0 \cup U_1)$. We have, $\lim \{y_s : s \in S\} \notin U_0 \cup U_1 \supset A$, contradicting the definition of the set A. Thus $A_m \subset U_0 \cup U_1$ for some $m \in M$. The facts $a_m \in U_0 \cap A_m$, $x_m \in U_1 \cap A_m$ and $U_0 \cap U_1 = \emptyset$ yield that the set A_m is not connected, a contradiction.
We have completed the proof that A is a closed connected subset of X. □

Proof of Theorem. Define a map $\phi : I^n \to \{0, \dots, n\}$ by

$$\phi(x) := \max \left\{ j : x \in \bigcap_{i=0}^{j} F_i^+ \right\}, \tag{1}$$

where $F_i^+ = I^n \times I$ and $F_i^+ = H_i^+ \backslash I_i^- \times I$ for each $i = 1, \ldots, n$. Since $I_i^\varepsilon \times I \subset H_i^\varepsilon$, where $\varepsilon = +$ or $-$, the map ϕ has the following properties:

$$\text{if } (x, t) \in I_i^- \times I, \text{ then } \phi(x, t) < i, \text{ and if } (x, t) \in I_i^+ \times I, \text{ then } \phi(x, t) \neq i - 1. \quad (2)$$

From (1) it follows that for each subset $S \subset I^n \times I$

$$\phi(S \cap I_i^\varepsilon \times I) = \{0, \ldots, n - 1\} \text{ implies that } i = n \text{ and } \varepsilon = - . \quad (3)$$

Observe that (2) and the fact that $I^n \times I = H_i^- \cup H_i^+$ imply that

$$\text{if } \phi(x) = i - 1 \text{ and } \phi(y) = i, \text{ then } x \in H_i^- \text{ and } y \in H_i^+ . \quad (4)$$

For each $k = 2, 3, \ldots$ the map $\phi \,|\, C(k) \times J(k)$ satisfies the condition (β) of Second Combinatorial Lemma and therefore there is a chain $S_0^k \sim \ldots \sim S_{m_k}^k$ of simplices such that

$$\phi(S_0^k \cap (C(k) \times \{0\})) = \{0, \ldots, n\} = \phi(S_{m_k}^k \cap (C(k) \times \{1\})).$$

Define connected sets

$$W_k := \bigcup_{i=0}^{m_k} \operatorname{conv} S_i^k, \qquad k = 2, 3, \ldots,$$

where conv A means the convex hull of the set A. Since $I^n \times I$ is a compact space we can find an infinite subset $M \subset N$ and convergent subsequence $\{w_m : m \in M\}$, $w_m \in W_m$. According to Lemma the set $W := \operatorname{Ls}\{W_m : m \in M\}$ is connected. Obviously

$$W \cap (I^n \times \{0\}) \neq \emptyset \neq W \cap (I^n \times \{1\}).$$

Let us prove that $W \subset \bigcap_{i=1}^n H_i^- \cap H_i^+$. To see this, fix $x \in W$ and choose a subsequence $\{x_k : k \in K\}, K \subset M$, of points $x_k \in W_k$ such that $\lim \{x_k : k \in K\} = x$. Next, choose n-colored $(n + 1)$-simplices S_k's, $S_k \subset W_k$, such that $x_k \in \operatorname{conv} S_k$. Since $\lim \operatorname{diam} \{\operatorname{conv} S_k : k \in K\} = 0$, we infer that for arbitrary subsequence $\{y_l : l \in L\}$, $L \subset K$, $y_l \in \operatorname{conv} S_l$, we have; $x = \lim \{y_l : l \in L\}$. Therefore the proof will be completed if we show that for each $i = 1, \ldots, n$ an n-colored $(n + 1)$-simplex S;

$$H_i^- \cap S \neq \emptyset \neq H_i^+ \cap S.$$

But it is clear in view of the property (4). □

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