Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 41 (2000), No. 2, 23--38

Persistent URL: http://dml.cz/dmlcz/702066

Some Recent Results on Cohen Algebras

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Received 11. March 2000

We supply some new proofs to recent results due to S. Koppelberg and S. Shelah concerning complete subalgebras of Cohen algebras.

1 Introduction

Let $\mathscr{R}_X$ be the Boolean algebra associated with the usual product measure on the generalized Cantor space $\{0, 1\}^X$. Maharam's Decomposition Theorem (cf. [F]) states that every measure algebra is a direct sum of countably many algebras $\langle \mathscr{R}_{X_n} : n < \omega \rangle$.

One can use the statement of Maharam's Theorem to define a notion of *category algebra.* Namely, let $\mathscr{C}_X$ be the Boolean algebra of regular open subsets of $\{0, 1\}^X$. We call $\mathscr{C}_X$ a *Cohen algebra.* A category algebra is then a direct sum of countably many Cohen algebras $\langle \mathscr{C}_{X_n} : n < \omega \rangle$.

The question that arises is whether a complete subalgebra of a category algebra is also a category algebra (this is obvious for measure algebras). In fact, this question quickly reduces to the following. Let $\mathscr{D}$ be a complete subalgebra of some Cohen algebra $\mathscr{C}_X$. Is then $\mathscr{D}$ a category algebra?

If X is countable then the Cantor's "back-and-forth" method gives the positive answer. It was shown by S. Koppelberg ([K]) that the answer is also "yes" when X is of cardinality ω_1. However, a recent result of S. Koppelberg and S. Shelah ([KS]) shows that we cannot go further: there exists (in ZFC) a subalgebra of $\mathscr{C}_{\omega_2}$ which is not a category algebra.

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1991 Mathemat cs Subject Classificatıon: 03E40, 03G05. Key words: Cohen algebras, complete subalgebras. Supported by KBN grant 2 P03A 011 09.

The present paper contains slightly simplified and modified presentation of these results.

2 Preliminaries

We will use the standard terminology and notation, compare e.g. [J]. In this section we recall some definitions; mainly to fix the notation.

2.1 Sets

Ordinal and cardinal numbers are in the sense of von Neumann. Thus ω is the set of natural numbers and ω_1 is the first uncountable cardinal. For arbitrary set X, let $\mathrm{Seq}(X)$ denote the set of all *finite* sequences (functions) s with $\mathrm{dom}(s) \subseteq X$ and $\mathrm{ran}(s) \subseteq \{0,1\}$. A set $A \subseteq \kappa$ is *unbounded* in κ, if $\forall \beta < \kappa \; \exists \alpha \in A \; \alpha > \beta$; otherwise A is *bounded.* A set $C \subseteq \kappa$ is *closed*, if $\sup B \in C$ for every bounded $B \subseteq A$. *Club* is a short for closed and unbounded.

2.2 Posets

Let $\mathscr{P}$, or more exactly $(\mathscr{P}, \leq)$, be a *poset* (partially ordered set). Two elements $p, q \in \mathscr{P}$ are *compatible*, if there exists $r \in \mathscr{P}$ such that $r \leq p$ and $r \leq q$; otherwise p, q are *incompatible* and we write $p \perp q$. For $p \in \mathscr{P}$ let $\mathscr{P}|p = \{q \in \mathscr{P} : q \leq p\}$ be the *reduced* poset. A set D is *dense* in $\mathscr{P}$, if $\forall p \in \mathscr{P} \; \exists q \in D \; q \leq p$. We also say that D is dense *below* p, if D is dense in $\mathscr{P}|p$. We say that $\mathscr{P}$ is *separative*, if $p \not\leq q$ implies the existence of $r \in \mathscr{P}$ such that $r \leq p$ but $r \perp q$. Finaly, $\mathscr{P}$ is *atomless* (or *splitting*), if $\forall p \in \mathscr{P} \; \exists q_0, q_1 \leq p$ such that $q_0 \perp q_1$.

2.3 Boolean algebras

Any Boolean algebra $\mathscr{B}$ may be treated as a poset when we define $a \leq b$ iff $a - b = \mathbf{0}$. We usually restrict $\leq$ to the set $\mathscr{B}^+ = \{b \in \mathscr{B} : b > \mathbf{0}\}$. Thus, when saying that $D \subseteq \mathscr{B}$ is dense in $\mathscr{B}$ we mean that for every $b \in \mathscr{B}^+$ there exists $q \in D$ such that $\mathbf{0} < q \leq b$. If $a > \mathbf{0}$ let $\mathscr{B}|a = \{b \in \mathscr{B} : b \leq a\}$ be the *reduced* Boolean algebra with the unit element a. By *density* of $\mathscr{B}$, denoted by $\partial(\mathscr{B})$, we mean the least cardinality of a dense subset of $\mathscr{B}$. Let us say that $\mathscr{B}$ has *uniform density* κ, if $\partial(\mathscr{B}|a) = \kappa$ for every $a \in \mathscr{B}^+$. By standard argument, every Boolean algebra can be decomposed into direct sum of algebras which have uniform density. If $A \subseteq \mathscr{B}$ we denote by $\sum A$ and $\prod A$ the least upper bound of A and the greatest lower bound of A, respectively (if they exist)[1]. If $\mathscr{B} < \mathscr{D}$ and $A \subset \mathscr{B}$ we use $\sum^{\mathscr{B}} A$ and $\sum^{\mathscr{D}} A$ to distinguish where this infinitary operations are computed. If $\mathscr{B} < \mathscr{D}$ are

[1] recall the convention: $\sum \emptyset = \mathbf{0}$ and $\prod \emptyset = \mathbf{1}$.

complete Boolean algebras and $\sum^{\mathscr{B}} A = \sum^{\mathscr{D}} A$ for every $A \subseteq \mathscr{B}$ then we call $\mathscr{B}$ a *complete subalgebra* of $\mathscr{D}$ and we write $\mathscr{B} <\circ \mathscr{D}$. If $\mathscr{B} < \mathscr{D}$ are complete then we have: $\mathscr{B} <\circ \mathscr{D}$ iff $\sum^{\mathscr{D}} A \in \mathscr{B}$ for every $A \subseteq \mathscr{B}$. A one-to-one homomorphism $e : \mathscr{B} \to \mathscr{D}$ is called a *complete embedding*, if $e(\sum A) = \sum\{e(a) : a \in A\}$ for every $A \subseteq \mathscr{B}$. Let us say that $G \subseteq \mathscr{D}$ *completely generates* $\mathscr{D}$, if the least complete subalgebra containing G is the whole $\mathscr{D}$.

If $\mathscr{B} <\circ \mathscr{D}$ then the *canonical projection* $\pi : \mathscr{D} \to \mathscr{B}$ is defined by the formula: $\pi(d) = \prod^{\mathscr{B}}\{b \in \mathscr{B} : b \geq d\}$. If $\mathscr{B} <\circ \mathscr{D}$ and $P \subseteq \mathscr{D}$ is dense in $\mathscr{D}$ then the set $\{\pi(d) : d \in P\} \subseteq \mathscr{B}$ is dense in $\mathscr{B}$. Thus $\partial(\mathscr{B}) \leq \partial(\mathscr{D})$ if $\mathscr{B} <\circ \mathscr{D}$. If $\mathscr{P}$ is a separative poset then there exists a unique complete Boolean algebra $\mathrm{RO}(\mathscr{P})$ such that $\mathscr{P}$ is (isomorphic to) a dense subset of $\mathrm{RO}(\mathscr{P})$. This algebra consists of all *regular open* subsets of $\mathscr{P}$ (see [J]). We shall make use of the following.

Lemma 2.1 *Let $\mathscr{P}$ and $\mathscr{Q}$ be separative posets and assume that $f : \mathscr{P} \to \mathscr{Q}$ is a function such that:*
1. $\mathrm{ran}(f) = \mathscr{Q}$*;*
2. if $p \leq q$ *then* $f(p) \leq f(q)$*;*
3. if $r \leq f(q)$ *then* $r = f(p)$ *for some* $p \leq q$*.*
Then the function e defined by the formula $e(R) = \{p \in \mathscr{P} : f(p) \in R\}$ *is a complete embedding* $e : \mathrm{RO}(\mathscr{Q}) \to \mathrm{RO}(\mathscr{P})$.

We call any above f a *neat cover*. We say that two poset $\mathscr{P}$ and $\mathscr{Q}$ are *equivalent*, if $\mathrm{RO}(\mathscr{P}) \cong \mathrm{RO}(\mathscr{Q})$. Any two countable atomless posets are equivalent. We say that $\mathscr{B}$ is *homogeneous*, if $\mathscr{B} \cong \mathscr{B}|a$ for every $a \in \mathscr{B}^+$. In this paper letters $\mathscr{B}$ and $\mathscr{D}$ will always denote Boolean algebras.

2.4 Cohen algebras

We have defined the Cohen algebra $\mathscr{C}_X$ as the Boolean algebra of regular open subsets of $\{0,1\}^X$. We can also define $\mathscr{C}_X$ as $\mathrm{RO}(\mathrm{Seq}(X))$, where $\mathrm{Seq}(X)$ is ordered by inverse inclusion: $s \leq t$ iff $s \supseteq t$. That is why we treat $\mathrm{Seq}(X)$ as a dense subset of $\mathscr{C}_X$. If $X \subseteq Y$ then the function $s \mapsto s|X$ is a neat cover from $\mathrm{Seq}(Y)$ to $\mathrm{Seq}(X)$. Thus we may treat $\mathscr{C}_X$ as a complete subalgebra of $\mathscr{C}_Y$. Finally, we can also define $\mathscr{C}_X$ as follows. Let $\mathrm{Clopen}(X)$ denote the family of sets that are simultaneously closed and open in $\{0,1\}^X$. Then $\mathrm{Clopen}(X)$ is dense in $\mathscr{C}_X$. Let $\mathscr{F}(X)$ be the σ-field generated by $\mathrm{Clopen}(X)$ and let $\mathscr{K}(X)$ be the σ-ideal of the sets of first category. Then $\mathscr{C}_X$ is $\mathscr{F}(X)/\mathscr{K}(X)$; which explains the word "category". It is easy to prove that $\mathscr{C}_X$ is homogenous for infinite X (note that $\mathrm{Seq}(X)$ is homogeneous). Let $\mathscr{C}$ denote the one-Cohen algebra $\mathscr{C}_\omega$.

2.5 Forcing

The Boolean value of a formula φ is denoted by $[\![\varphi]\!]$. We write $p \Vdash \varphi$ (*p forces* φ) iff $p \leq [\![\varphi]\!]$. When it is possible we adopt the convention of identifying the object from the generic extension with its name. For other definitions, like generic filter, minimal product, iteration etc, we refer to [J].

3 Independence

Definition. Let $\mathscr{B} < \mathscr{D}$ and $u \in \mathscr{D}$. We say that u is *independent over* $\mathscr{B}$, if for every $b \in \mathscr{B}^+$ we have $b \cdot u > \mathbf{0}$ and $b \cdot (-u) > \mathbf{0}$. In other words u splits every $b \in \mathscr{B}^+$.

Let $\{b_i : i \in X\}$ be an indexed family of elements of $\mathscr{B}$. For the purpose of the following definition, if $s \in \operatorname{Seq}(X)$ then we let

$$b_s = \prod_{\substack{i \in \operatorname{dom}(s) \\ s(i) = 1}} b_i \cdot \prod_{\substack{i \in \operatorname{dom}(s) \\ s(i) = 0}} (-b_i).$$

Definition. We say that a family $I = \{b_i : i \in X\} \subseteq \mathscr{B}$ is *independent*, if $b_s > \mathbf{0}$ for every $s \in \operatorname{Seq}(X)$. We also say that I is $*$-dense in $\mathscr{B}$, if the set $\{b_s : s \in \operatorname{Seq}(X)\}$ is dense in $\mathscr{B}$.

Note the following connection: if $I \subseteq \mathscr{B}$ is independent and u is independent over $\mathscr{B}$ then $I \cup \{u\}$ is independent.

Using the above notions we obtain the following simple characterization of Cohen algebras.

Lemma 3.1 *Let $\mathscr{D}$ be a complete Boolean algebra and $\kappa \geq \omega$. The following are equivalent:*

1. $\mathscr{D} \cong \mathscr{C}_\kappa$;
2. *There exist $I \subseteq \mathscr{D}$ of cardinality κ which is independent and $*$-dense in $\mathscr{D}$.*

Proof. ($1 \to 2$) For $\alpha < \kappa$ let $b_\alpha \in \operatorname{Seq}(\kappa)$ be such that $\operatorname{dom}(b_\alpha) = \{\alpha\}$ and $b_\alpha(\alpha) = 1$. Then $b_s = s$ for $s \in \operatorname{Seq}(\kappa)$. Therefore $\{b_\alpha : \alpha < \kappa\}$ is independent and $*$-dense in $\mathscr{C}_\kappa$.

($2 \to 1$) Write $I = \{b_\alpha : \alpha < \kappa\}$ and consider the dense set $\mathscr{P} = \{b_s : s \in \operatorname{Seq}(\kappa)\}$. Then $(\mathscr{P}, \leq) \cong (\operatorname{Seq}(\kappa), \supseteq)$ and hence $\mathscr{C}_\kappa \cong \operatorname{RO}(\mathscr{P}) \cong \mathscr{D}$. $\square$

Therefore, to show that some complete Boolean algebra $\mathscr{D}$ is Cohen, it suffices to find an independent $*$-dense subset of $\mathscr{D}$. Such a subset will be usually constructed by transfinite induction. In the next section we show how to handle a single step extension.

4 Single extension of Boolean algebra

Let $\mathscr{B} < \mathscr{D}$ and $u \in \mathscr{D}$. Let $\mathscr{B}(u)$ denote the smallest subalgebra of $\mathscr{D}$ containing $\mathscr{B}$ and u. It is easy to check that actually

$$\mathscr{B}(u) = \{a \cdot u + b \cdot (-u) : a, b \in \mathscr{B}\}.$$

Lemma 4.1 *If $\mathscr{B} <\circ \mathscr{D}$ and $u \in \mathscr{D}$ then $\mathscr{B}(u) <\circ \mathscr{D}$.*

Proof. Let $A \subseteq \mathscr{B}(u)$. It suffices to show that $\sum^{\mathscr{D}} A \in \mathscr{B}(u)$. Write

$$A = \{a_\alpha \cdot u + b_\alpha \cdot (-u) : \alpha < \kappa\}$$

where $\{a_\alpha\}, \{b_\alpha\} \subseteq \mathscr{B}$ and note that (computing everything in $\mathscr{D}$)

$$\sum A = \sum_{\alpha<\kappa} a_\alpha \cdot u + b_\alpha \cdot (-u) = \left(\sum_{\alpha<\kappa} a_\alpha\right) \cdot u + \left(\sum_{\alpha<\kappa} b_\alpha\right) \cdot (-u)$$

But the elements in the large parenthesis are in $\mathscr{B}$ because $\mathscr{B} <\circ \mathscr{D}$. $\square$

Corollary 4.2 *If $\mathscr{B} <\circ \mathscr{D}$ and $u \in \mathscr{D}$ then $\mathscr{B} <\circ \mathscr{B}(u)$.*

Proof. Let $A \subseteq \mathscr{B}$. We must show that $\sum^{\mathscr{B}(u)} A \in \mathscr{B}$. But $\sum^{\mathscr{B}(u)} A = \sum^{\mathscr{D}} A$ by Lemma 4.1. And $\sum^{\mathscr{D}} A \in \mathscr{B}$ because $\mathscr{B} <\circ \mathscr{D}$. $\square$

Consider now the following situation. Let $\mathscr{B} < \mathscr{D}$, $u \in \mathscr{D}$ and let P be a dense subset of $\mathscr{B}$. Put

$$Q = \{p \cdot u, p \cdot (-u) : p \in P\}.$$

It is natural to expect that Q will be dense in $\mathscr{B}(u)$. However, this is not true in general. For example, if $\sum^{\mathscr{D}} P = v < 1$ and we let $u = -v$ then for no $\mathbf{0} < q \in Q$ we have $q \leq u$. Nevertheless we have the following.

Lemma 4.3 *If $\mathscr{B} <\circ \mathscr{D}$ or if u is independent over $\mathscr{B}$ then Q is dense in $\mathscr{B}(u)$.*

Proof. Easy exercise. $\square$

Corollary 4.4 *Let $\mathscr{B} < \mathscr{D}$ and assume that $I \subseteq \mathscr{B}$ is independent and $*$-dense in $\mathscr{B}$. Let $u \in \mathscr{D}$ be independent over $\mathscr{B}$. Then $I \cup \{u\}$ is independent and $*$-dense in $\mathscr{B}(u)$.*

The above corollary shows that in order to extend a given independent and $*$-dense set, we need an independent element. The fairly general construction of such an element is described in the following lemma.

Lemma 4.5 (Vladimirov) *Assume that $\mathscr{B} <\circ \mathscr{D}$ and $\mathscr{B}|b \neq \mathscr{D}|b$ for every $b \in \mathscr{B}^+$. Then, for every $x \in \mathscr{D}$ there exists $u \in \mathscr{D}$ such that u is independent over $\mathscr{B}$ and $x \in \mathscr{B}(u)$.*

Proof. Let $\pi : \mathscr{D} \to \mathscr{B}$ be the canonical projection and let $\mathrm{ind}(d) = \pi(d) \cdot \pi(-d)$. Then $\mathrm{ind}(d) \in \mathscr{B}$ and we have: $\mathrm{ind}(d) > \mathbf{0}$ iff $d \notin \mathscr{B}$. Moreover, if $b \in \mathscr{B}$ and $b \cdot \mathrm{ind}(d) > \mathbf{0}$ then $b \cdot d > \mathbf{0}$ and $b \cdot (-d) > \mathbf{0}$.

CLAIM: *The set $\{\mathrm{ind}(d) : d \notin \mathscr{B}\}$ is dense in $\mathscr{B}$.*

PROOF: Fix $b \in \mathscr{B}^+$. Then $\mathscr{B}|b \neq \mathscr{D}|b$. Hence, there exists $d \leq b$ such that $d \notin \mathscr{B}$. But then $\mathbf{0} < \text{ind}(d) \leq \pi(d) \leq b$. $\square$ (of CLAIM)

We can assume that $\text{ind}(x) > \mathbf{0}$ (otherwise replace x by any element not in $\mathscr{B}$). Using CLAIM it is easy to build a partition $\{\text{ind}(d_\alpha) : \alpha < \kappa\}$ of $\mathscr{B}$ such that $d_0 = x$, $d_\alpha \notin \mathscr{B}$ and for $\alpha > 0$ either $\text{ind}(d_\alpha) \cdot \pi(x) = \mathbf{0}$ or $\text{ind}(d_\alpha) \cdot \pi(-x) = \mathbf{0}$. Now define

$$u = \sum_{\alpha<\kappa} \text{ind}(d_\alpha) \cdot d_\alpha$$

and note that

$$-u = \sum_{\alpha<\kappa} \text{ind}(d_\alpha) \cdot (-d_\alpha).$$

To prove that u is independent over $\mathscr{B}$ consider any $b \in \mathscr{B}^+$. Then $b \cdot \text{ind}(d_\alpha) > \mathbf{0}$ for some $\alpha < \kappa$. We can assume that actually $b \leq \text{ind}(d_\alpha)$. But then $\mathbf{0} < b \cdot d_\alpha \leq b \cdot u$ and $\mathbf{0} < b \cdot (-d_\alpha) \leq b \cdot (-u\}$ as required. Finally, we must show that $x \in \mathscr{B}(u)$. We can write

$$x = \sum_{\alpha<\kappa} \text{ind}(d_\alpha) \cdot x$$

because, by $\mathscr{B} <\circ \mathscr{D}$, our family is also a partition of $\mathscr{D}$. Thus, by Lemma 4.1 it suffices to show that $\text{ind}(d_\alpha) \cdot x \in \mathscr{B}(u)$ for every $\alpha < \kappa$. We have $\text{ind}(d_0) \cdot x = \text{ind}(d_0) \cdot u \in \mathscr{B}(u)$. If $\alpha > 0$ then either $\text{ind}(d_0) \cdot x = \mathbf{0}$ or $\text{ind}(d_\alpha) \leq x$. In both cases $\text{ind}(d_\alpha) \cdot x \in \mathscr{B}(u)$. $\square$

Let us introduce more notation. Write $\mathscr{B}(u, v)$ for $\mathscr{B}(u)(v)$ and more generally $\mathscr{B}(u_k : k \leq n) = \mathscr{B}(u_k : k < n)(u_n)$. Also, if $\mathscr{B} < \mathscr{D}$ and $\{u_n : n < \omega\} \subseteq \mathscr{D}$ let

$$\mathscr{B}(u_n : n < \omega) = \bigcup_n \mathscr{B}(u_k : k < n).$$

5 Koppelberg's Theorem

Now we are ready to prove Koppelberg's Theorem.

Theorem 5.1 *If $\mathscr{D} <\circ \mathscr{C}_{\omega_1}$ and $\mathscr{D}$ has uniform density ω_1 then $\mathscr{D} \cong \mathscr{C}_{\omega_1}$.*

Proof. Let $\mathscr{B}_\alpha = \mathscr{D} \cap \mathscr{C}_\alpha$ for $\alpha < \omega_1$. Then $\mathscr{B}_\alpha$'s form an increasing chain of complete subalgebras of $\mathscr{D}$ and $\mathscr{D} = \bigcup_{\alpha<\omega_1} \mathscr{B}_\alpha$. Also $\mathscr{B}_\alpha <\circ \mathscr{C}_\alpha$ and therefore $\partial(\mathscr{B}_\alpha) \leq \omega$.

CLAIM: *There is a club set $C \subseteq \omega_1$ such that $\bigcup_{\beta<\alpha} \mathscr{B}_\beta$ is dense in $\mathscr{B}_\alpha$ whenever $\alpha \in C$.*

PROOF: Let $\pi : \mathscr{C}_{\omega_1} \to \mathscr{D}$ be the canonical projection. Put

$$A = \{\beta < \omega_1 : \pi(s) \in \mathscr{B}_\beta \text{ for every } s \in \text{Seq}(\beta)\}$$

A simple closure argument shows that A is club in ω_1. Let C be the set of all limit points of A. Then C is also club.

Let $\alpha \in C$ and fix $b \in \mathscr{B}_\alpha^+$. Then $b \in \mathscr{C}_\alpha^+$ and therefore there exists $s \in \mathrm{Seq}(\alpha)$ such that $s \leq b$. There must be some $\beta < \alpha$ such that $\beta \in A$ and $s \in \mathrm{Seq}(\beta)$. But then $\pi(s) \in \mathscr{B}_\beta$ and $\mathbf{0} < \pi(s) \leq b$. □ (of CLAIM)

If $\alpha > \beta$ let us say that $\mathscr{B}_\alpha$ is a *good* extension of $\mathscr{B}_\beta$, if there exists a sequence $\{u_n : n < \omega\} \subseteq \mathscr{B}_\alpha$ such that every u_n is independent over $\mathscr{B}_\beta(u_k : k < n)$ and $\mathscr{B}_\beta(u_n : n < \omega)$ is dense in $\mathscr{B}_\alpha$. Note the following: if $I \leq \mathscr{B}_\beta$ is independent and $*$-dense in $\mathscr{B}_\beta$ then $I \cup \{u_n : n < \omega\}$ is independent and $*$-dense in $\mathscr{B}_\alpha$. The next fact is the heart of the proof.

Lemma 5.2 *For every $\beta < \omega_1$ the set*

$$G_\beta = \{\alpha > \beta : \alpha \in C \text{ and } \mathscr{B}_\alpha \text{ is a good extension of } \mathscr{B}_\beta\}$$

is unbounded in ω_1.

Proof. Fix $\gamma > \beta$. We shall find $\alpha > \gamma$ such that $\alpha \in G_\beta$. We construct inductively a sequence $\alpha_0 < \alpha_1 < \ldots$ in C. At the n-th step we will use Vladimirov's Lemma 4.5 to find an independent element u_n. The elements x that we want to include in a single extensions will be listed in the array $\{x_m^n : m, n < \omega\}$. At step n we define the n-th row of this array. To handle the entire array in ω steps we need a pairing function $n \mapsto \langle f(n), g(n)\rangle$ such that $f(n) \leq n$. To start, let $\{x_m^0 : m < \omega\}$ be a dense subset of $\mathscr{B}_\gamma$. Let $u_0 \in \mathscr{D}$ be independent over $\mathscr{B}_\beta$ and such that $x_{g(0)}^{f(0)} \in \mathscr{B}_\beta(u_0)$. Now find $\alpha_0 \in C$ such that $\alpha_0 > \gamma$ and $\mathscr{B}_\beta(u_0) \subseteq \mathscr{B}_{\alpha_0}$. Generally, at step n, let $\{x_m^n : m < \omega\}$ be a dense subset of $\mathscr{B}_{\alpha_{n-1}}$ and find $u_n \in \mathscr{D}$ independent over $\mathscr{B}_\beta(u_k : k < n)$ such that $x_{g(n)}^{f(n)} \in \mathscr{B}_\beta(u_k : k \leq n)$. Then pick $\alpha_n > \alpha_{n-1}$ such that $\alpha_n \in C$ and $\mathscr{B}_\beta(u_k : k \leq n) \subseteq \mathscr{B}_{\alpha_n}$. This completes the inductive definition. Notice that Vladimirov's Lemma is applicable here because $\partial(\mathscr{B}_\beta(u_k : k < n)) \leq \omega$ and $\mathscr{D}$ has uniform density ω_1. Put $\alpha = \sup_n \alpha_n$. Then $\alpha \in C$ and $\alpha > \gamma$. By construction $\mathscr{B}_\beta(u_n : n < \omega)$ contains the set $\{x_m^n : m, n < \omega\}$ which is dense in $\bigcup_{\beta<\alpha}\mathscr{B}_\beta$. But this last union is dense in $\mathscr{B}_\alpha$ because $\alpha \in C$. Therefore $\mathscr{B}_\alpha$ is a good extension of $\mathscr{B}_\beta$. □ (of Lemma 5.2)

Now we finish the proof of Koppelberg's Theorem. Starting from $\mathscr{B}_0 = \{\mathbf{0}, \mathbf{1}\}$ and using Lemma 5.2 build a cofinal sequence $\langle \mathscr{B}_{\alpha_\eta} : \eta < \omega_1\rangle$, where $\alpha_\eta \in C$, together with $I_\eta \subseteq \mathscr{B}_{\alpha_\eta}$ independent and $*$-dense in $\mathscr{B}_{\alpha_\eta}$. Then $I = \bigcup_{\eta<\omega_1} I_\eta$ is independent and $*$-dense in D. Moreover I has cardinality ω_1. Hence $\mathscr{D} \cong \mathscr{C}_{\omega_1}$ by Lemma 3.1. □

Corollary 5.3 *If $\mathscr{D} <\circ\, \mathscr{C}_{\omega_1}$ then $\mathscr{D}$ is a category algebra.*

Proof. Decompose $\mathscr{D}$ into direct sum of algebras with uniform density. □

6 The Boolean algebra $\mathrm{RO}(Q_X)$

The rest of the paper is devoted to example, due to S. Koppelberg and S. Shelah ([KS]), of Boolean algebra which is not Cohen but can be completely embedded into a Cohen algebra. We start with the following useful definition.

Definition. If $m \le n \le \omega$ and $A \subseteq \omega^n$, let us say that A is *disjoint above m*, if for any distinct $s, t \in A$ we have $s(i) \neq t(i)$ for all i such that $m \le i < n$.

The main object of our study is the Boolean algebra $\mathrm{RO}(Q_X)$, where Q_X is the following poset.

Definition. Let X be a nonempty set. The elements of Q_X are finite functions p, such that $\mathrm{dom}(p) \subseteq X$ and $\mathrm{ran}(p) \subseteq \omega^n$ for some $1 \le n < \omega$. If $p \neq \emptyset$ then the above n is unique. We denote it by $\mathrm{ht}(p)$ and we call it *the height of p*. Let also $\mathrm{ht}(\emptyset) = 0$. For $p, q \in Q_X$ write $p \le q$ iff

1. $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$;
2. $p(\alpha) \supseteq q(\alpha)$ for all $\alpha \in \mathrm{dom}(q)$;
3. $\{p(\alpha) : \alpha \in \mathrm{dom}(q)\}$ is disjoint above $\mathrm{ht}(q)$.

Then $(Q_X, \le)$ is a poset with the greatest element $\emptyset$. Note that $p \le q$ implies $\mathrm{ht}(p) \ge \mathrm{ht}(q)$.

Lemma 6.1 *Assume that $p, q \in Q_X$ and $\mathrm{ht}(p) \ge \mathrm{ht}(q)$. Then p, q are compatible iff*

1. $p(\alpha) \supseteq q(\alpha)$ for all $\alpha \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$ and

2. $\{p(\alpha) : \alpha \in \mathrm{dom}(p) \cap \mathrm{dom}(q)\}$ is disjoint above $\mathrm{ht}(q)$.

Proof. Easy exercise. □

For completeness sake we prove the following lemma (cf. [KS]).

Lemma 6.2 *Q_X is separative.*

Proof. Let $p, q \in Q_X$ and $p \not\le q$. Then $q \neq \emptyset$. We shall find $r \le p$ such that $r \perp q$. We may assume that p, q are compatible (otherwise let $r = p$). Consider three cases.

1. $\mathrm{dom}(p) \not\supseteq \mathrm{dom}(q)$. Pick $\alpha \in \mathrm{dom}(q) \setminus \mathrm{dom}(p)$ and let $r \le p$ be such that $\mathrm{dom}(r) = \mathrm{dom}(p) \cup \{\alpha\}, \mathrm{ht}(r) = \mathrm{ht}(p) + 1$ and $r(\alpha)(0) \neq q(\alpha)(0)$.
2. $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and $\mathrm{ht}(p) \ge \mathrm{ht}(q)$. This case is impossible by Lemma 6.1.
3. $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and $\mathrm{ht}(p) < \mathrm{ht}(q)$. Fix any $\alpha \in \mathrm{dom}(q)$. Let $r \le p$ be such that $\mathrm{dom}(r) = \mathrm{dom}(p)$, $\mathrm{ht}(r) = \mathrm{ht}(q)$ but $r(\alpha)(\mathrm{ht}(p)) \neq q(\alpha)(\mathrm{ht}(p))$. □

Therefore Q_X may be treated as a dense subset of $\mathrm{RO}(Q_X)$. Note that $|Q_X| = |X| + \omega$. Thus Q_1, Q_2 and Q_ω are all equivalent to one-Cohen forcing $\mathscr{C}$. We also leave to the reader the verification that Q_X satisfies c.c.c..

Definition. $D_{Y,m} = \{p \in Q_X : Y \subseteq \mathrm{dom}(p) \text{ and } \mathrm{ht}(p) > m\}$.

Note that $D_{Y,m}$ is dense in Q_X for every finite $Y \subseteq X$ and $m < \omega$.

Lemma 6.3. *Assume that $\emptyset \neq Y \subseteq X$. The function $p \mapsto p|Y$ from Q_X onto Q_Y is a neat cover.*

Proof. Easy exercise. $\square$

Therefore, the formula $e(R) = \{p \in Q_X : p|Y \in R\}$ defines a complete embedding $e : \mathrm{RO}(Q_Y) \to \mathrm{RO}(Q_X)$. By separativity, we may identify $q \in Q_Y$ with the set $\{r \in Q_Y : r \leq q\}$. Then $e(q) = \{p \in Q_X : p|Y \leq q\}$. But note that for $p \in Q_X$ we have: $p|Y \leq q$ (in Q_Y) iff $p \leq q$ (in Q_X). Thus we may think that $e(q) = q$ and that $\mathrm{RO}(Q_Y) <\circ \mathrm{RO}(Q_X)$.

Consider the special case when $Y = \{\alpha\}$. Nonempty elements of $Q_{\{\alpha\}}$ are of the form $\{\langle \alpha, s\rangle\}$ where $s \in \omega^n$ for some $n \geq 1$. Let $\mathrm{dom}(p) = \{\alpha, \beta\}$ and $\alpha \neq \beta$. Then, unlike in the minimal product, we have $p \neq \{\langle \alpha, p(\alpha)\rangle\} \cdot \{\langle \beta, p(\beta)\rangle\}$. To see this extend $p(\alpha)$ and $p(\beta)$ by the *same* value. This will be generalized in Lemma 6.6.

Lemma 6.4. $\mathrm{RO}(Q_X)$ *is completely generated by the set* $\bigcup\{Q_{\{\alpha\}} : \alpha \in X\}$.

Proof. Q_X is dense in $\mathrm{RO}(Q_X)$. Thus if $b \in \mathrm{RO}(Q_X)$ then $b = \sum\{p \in Q_X : p \leq b\}$. Hence, it suffices to show that every element $p \in Q_X$ is generated by elements from $\bigcup\{Q_{\{\alpha\}} : \alpha \in Y\}$ where $Y = \mathrm{dom}(p)$. We show that actually:

$$(*) \qquad p = \prod_{n<\omega} \sum_{\substack{q \leq p \\ \mathrm{ht}(q) > n}} \prod_{\alpha \in Y} \{\langle \alpha, q(\alpha)\rangle\}$$

First note that $q \leq \{\langle \alpha, q(\alpha)\rangle\}$ if $\alpha \in \mathrm{dom}(q)$. Therefore $q \leq \prod_{\alpha \in Y}\{\langle \alpha, q(\alpha)\rangle\}$ if $q \leq p$. Thus, for every $n < \omega$

$$\sum_{\substack{q \leq p \\ \mathrm{ht}(q) > n}} \prod_{\alpha \in Y} \{\langle \alpha, q(\alpha)\rangle\} \geq \sum_{\substack{q \leq p \\ \mathrm{ht}(q) > n}} \prod_{\alpha \in Y} q = p$$

For the last equality note that $D_{Y,n}$ is dense in Q_X. Hence in $(*)$ we have RHS $\geq p$.

Now let $r \leq$ RHS. We show that $r \leq p$. We can assume that r belongs to the dense set $D_{Y,\mathrm{ht}(p)}$. Let $n = \mathrm{ht}(r)$. Then for some $q \leq p$ with $\mathrm{ht}(q) > n$ our r is compatible with every $\{\langle \alpha, q(\alpha)\rangle\}$ for $\alpha \in Y$. It follows that $r(\alpha) \subseteq q(\alpha)$ for $\alpha \in Y$. But $q \leq p$. Hence $\{r(\alpha) : \alpha \in Y\}$ is disjoint above $\mathrm{ht}(p)$. So $r \leq p$ as required. $\square$

Corollary 6.5. $\mathrm{RO}(Q_{X \cup Y})$ *is completely generated by* $Q_X \cup Q_Y$.

The following lemma describes the crucial property of Q_X.

Lemma 6.6. *Assume that p, q are compatible in Q_X. Assume also that $\alpha \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$ and $\beta \in \mathrm{dom}(q) \setminus \mathrm{dom}(p)$. Let $d \in Q_X$ be such that $\mathrm{dom}(d) = \{\alpha, \beta\}$ and $\mathrm{ht}(d) \leq \min\{\mathrm{ht}(p), \mathrm{ht}(q)\}$. Then there exists $r \in Q_X$ such that $r \leq p, q$ and $r \perp d$ i.e., $p \cdot q \not\leq d$ in $\mathrm{RO}(Q_X)$.*

Proof. We can assume that $p(\alpha) \supseteq d(\alpha)$ and $q(\beta) \supseteq d(\beta)$. Otherwise $p \cdot q \cdot d = 0$ so $p \cdot q \not\leq d$. Also, using Lemma 6.1 and compatibility of p, q we can assume that $\mathrm{ht}(p) = \mathrm{ht}(q) = m$. Now, from assumptions about α and β it is possible to define

r with $\mathrm{ht}(r) = m + 1$ such that $r \le p$, $r \le q$ and $r(\alpha)(m) = r(\beta)(m)$. Then r is incompatible with d. $\square$

Proposition 6.7. *If $|X| \ge \omega_2$ then $\mathrm{RO}(Q_X)$ is not a category algebra.*

Proof. By contradiction. Assume that $\mathrm{RO}(Q_X)$ is a category algebra. Observe that $\mathrm{RO}(Q_X)$ has uniform density $\kappa = |X|$. Thus, in the direct sum decomposition all components must have density equal to $|X|$. We can therefore assume that $\mathrm{RO}(Q_X) \cong \mathscr{C}_X$. Hence, we may treat Q_X as a dense subset of $\mathscr{C}_X$. But $\mathscr{C}_X$ has also a standard dense set $\mathrm{Seq}(X)$.

Let us call $Y \subseteq X$ *admissible*, if Q_Y is dense in $\mathscr{C}_Y$. An easy closure argument shows that for every $Y \subseteq X$ there is an admissible $Y^* \supseteq Y$ such that $|Y^*| \le |Y| + \omega$.

Now fix $Y \subseteq X$ such that $|Y| = \omega_1$ and pick $\alpha \in X \setminus Y^*$. Here we use the assumption that $|X| \ge \omega_2$. Because $|\{\alpha\}^*| \le \omega$ we can pick $\beta \in Y \setminus \{\alpha\}^*$.

Fix also arbitrary $d \in Q_X$ such that $\mathrm{dom}(d) = \{\alpha, \beta\}$. Then $d \in Q_{\{\alpha,\beta\}}$.

But $\mathrm{RO}(Q_{\{\alpha,\beta\}})$ is completely generated by $Q_{\{\alpha\}} \cup Q_{\{\beta\}}$. Also $Q_{\{\alpha\}} \subseteq \mathscr{C}_{\{\alpha\}^*}$ and $Q_{\{\beta\}} \subseteq \mathscr{C}_{Y^*}$. Therefore $d \in \mathscr{C}_{\{\alpha\}^* \cup Y^*}$. But then there is some $t \in \mathrm{Seq}(\{\alpha\}^* \cup Y^*)$ such that $t \le d$. We can write $t = a \cdot b$ where $a \in \mathscr{C}_{\{\alpha\}^*}$ and $b \in \mathscr{C}_{Y^*}$. By admissibility of $\{\alpha\}^*$ the element a can be written as a Boolean infinite sum of elements p from $Q_{\{\alpha\}^*}$ such that $a \in \mathrm{dom}(p)$ and $\mathrm{ht}(p) > \mathrm{ht}(d)$. Similarly for b. Hence we can find $p \in Q_{\{\alpha\}^*}$ and $q \in Q_{Y^*}$ such that $\mathrm{ht}(p), \mathrm{ht}(q) > \mathrm{ht}(d)$, $\alpha \in \mathrm{dom}(p) \subseteq \{\alpha\}^*$, $\beta \in \mathrm{dom}(q) \subseteq Y^*$ and $\mathbf{0} < p \cdot q \le d$. But this contradicts the Lemma 6.6. $\square$

7 Embedding $\mathrm{RO}(Q_X)$ into $\mathscr{C}_X$

If X is countable then Q_X is equivalent to one-Cohen forcing $\mathscr{C}$. Here we assume that X is infinite and we show that $\mathrm{RO}(Q_X)$ can be completely embedded into $\mathscr{C}_X$. The idea is to show that C_X adjoins a *generic filter* (cf. [J]) on Q_X. Then we use the following folklore lemma.

Lemma 7.1. *Assume that $\mathscr{Q}$ is a separative poset and $\mathscr{B}$ is a complete Boolean algebra. Assume also that there exists a $\mathscr{B}$-valued name $\mathscr{G}$ with properties:*

1. *$[\![\mathscr{G}$ is a generic filter on $\mathscr{Q}]\!] = \mathbf{1}$;*
2. *$[\![p \in \mathscr{G}]\!] > \mathbf{0}$ for every $p \in \mathscr{Q}$.*

Then the function e defined by the formula $e(R) = \sum\{[\![p \in \mathscr{G}]\!] : p \in R\}$ is a complete embedding $e : \mathrm{RO}(\mathscr{G}) \to \mathscr{B}$.

First we describe generic filters on Q_X. In the following we fix a transitive model $\mathscr{M}$ of ZFC such that $Q_X \in \mathscr{M}$. Let $\mathscr{G}$ be an $\mathscr{M}$-generic filter on Q_X. In the generic extension $\mathscr{M}[\mathscr{G}]$ define the sequence $\langle f_\alpha : \alpha \in X \rangle \subseteq \omega^\omega$ by writting

$$f_\alpha(n) = m \text{ iff } p(\alpha)(n) = m \text{ for some } p \in \mathscr{G} \text{ with } \mathrm{ht}(p) > n \text{ and } \alpha \in \mathrm{dom}(p).$$

The density argument shows that $\langle f_\alpha : \alpha \in X \rangle$ is an almost disjoint family. Note that every f_α is a Cohen real over $\mathscr{M}$. Hovewer, if $\alpha \neq \beta$ then f_α is not a Cohen real over $\mathscr{M}[f_\beta]$. Namely, $f_\alpha \in \{f \in \omega^\omega : f$ and f_β are almost disjoint$\}$. But this last set is of first category and is coded in $\mathscr{M}[f_\beta]$.

Now we show how an arbitrary sequence $\langle f_\alpha : \alpha \in X \rangle$ gives rise to a certain filter (not necessarily generic) on Q_X.

Definition. Let $\langle f_\alpha : \alpha \in X \rangle \subseteq \omega^\omega$. We shall say that $p \in Q_X$ is *coherent* with $\langle f_\alpha : \alpha \in X \rangle$, if

1. $f_\alpha \supseteq p(\alpha)$ for all $\alpha \in \mathrm{dom}(p)$;
2. the set $\{f_\alpha : \alpha \in \mathrm{dom}(p)\}$ is disjoint above $\mathrm{ht}(p)$.

Note that p is coherent with $\langle f_\alpha : \alpha \in X \rangle$ iff p is coherent with $\langle f_\alpha : \alpha \in \mathrm{dom}(p) \rangle$.

Definition. $\mathscr{G}\langle f_\alpha : \alpha \in X \rangle = \{p \in Q_X : p$ is coherent with $\langle f_\alpha : \alpha \in X \rangle\}$.

Lemma 7.2. *If $\langle f_\alpha : \alpha \in X \rangle \subseteq \omega^\omega$ is almost disjoint then $\mathscr{G}\langle f_\alpha : \alpha \in X \rangle$ is a filter on Q_X.*

Proof. $\mathscr{G}\langle f_\alpha : \alpha \in X \rangle$ contains $\emptyset$ and is upward-closed. If $p, q \in \mathscr{G}\langle f_a : a \in X \rangle$ let $m > \mathrm{ht}(p), \mathrm{ht}(q)$ be such that $\{f_\alpha : \alpha \in \mathrm{dom}(p) \cup \mathrm{dom}(q)\}$ is disjoint above m. Define $r \in Q_X$ by: $\mathrm{dom}(r) = \mathrm{dom}(p) \cup \mathrm{dom}(q)$ and $r(\alpha) = f_\alpha | m$. Then $r \leq p, q$ and $r \in \mathscr{G}\langle f_\alpha : \alpha \in X \rangle$. $\square$

Definition. Let us call $\langle f_\alpha : \alpha \in X \rangle \subseteq \omega^\omega$ and *$\mathscr{M}$-generic sequence* on Q_X, if it is almost disjoint and $\mathscr{G}\langle f_a : a \in X \rangle$ is an $\mathscr{M}$-generic filter on Q_X.

We need also a weaker notion.

Definition. Let us say that $\langle f_\alpha : \alpha \in X \rangle \subseteq \omega^\omega$ is a *weak $\mathscr{M}$*-generic sequence on Q_X, if $\langle f_\alpha : \alpha \in Y \rangle$ is an $\mathscr{M}$-generic sequence on Q_Y for every *finite* nonempty $Y \subseteq X$.

Surprisingly, a large weak $\mathscr{M}$-generic sequence exists in one-Cohen extension of $\mathscr{M}$.

Lemma 7.3. *Assume that $|X| \leq 2^\omega$ in $\mathscr{M}$. Then in one-Cohen extension $\mathscr{M}[c]$ there exists a weak $\mathscr{M}$-generic sequence on Q_X.*

Proof. Fix in $\mathscr{M}$ a sequence $\langle f_\alpha : \alpha \in X \rangle \subseteq \omega^\omega$ of distinct functions. This is possible because $|X| \leq 2^\omega$. Consider the following separative poset $\mathscr{P}$. The elements of $\mathscr{P}$ are finite functions f with $\mathrm{dom}(f) \subseteq \bigcup_n \omega^n$ and $\mathrm{ran}(f) \subseteq \omega$ such that $f | \omega^n$ is one-to-one for every $n < \omega$. The order is extension: $f \leq g$ iff $f \supseteq g$.

Of course $\mathscr{P}$ is equivalent to one-Cohen forcing and $\mathscr{P}$ adjoins a function $F : \bigcup_n \omega^n \to \omega$ such that $F | \omega^n$ is one-to-one for every n. Note that $[\![f \subseteq F]\!] = f$ for $f \in \mathscr{P}$.

In $\mathscr{M}[F]$ put $f_\alpha(n) = F(y_\alpha|n)$ for $n > 0$ and $\alpha \in X$. Here, for notational convenience, we define f_α's on $\omega \setminus \{0\}$.

We claim that $\langle f_\alpha : \alpha \in X \rangle$ is weak $\mathscr{M}$-generic sequence on Q_X. So fix a finite $Y \subseteq X$ and let D be a dense subset of Q_Y from $\mathscr{M}$. Let $f \in \mathscr{P}$ be arbitrary. We shall find $g \leq f$ and $p \in D$ such that

$$g \Vdash p \text{ is coherent with } \langle f_\alpha : \alpha \in Y \rangle .$$

Let m be large enough such that $\mathrm{dom}(f) \subseteq \omega^{<m}$ and $y_\alpha|m$ are all distinct for $\alpha \in Y$. By extending f if necessary we can additionally assume that $\{y_\alpha|n : n < m$ and $\alpha \in Y\} \subseteq \mathrm{dom}(f)$. Then f decides $\langle f_\alpha|m : \alpha \in Y \rangle$ because

$$f \Vdash f_\alpha(n) = f(y_\alpha|n) \text{ for } n < m .$$

Let $q \in Q_Y$ be such that $\mathrm{dom}(q) = Y$ and $q(\alpha) = f_\alpha|m$. As D is dense, there exists $p \leq q$ such that $p \in D$. Now $\{p(\alpha) : \alpha \in Y\}$ is disjoint above $\mathrm{ht}(q) = m$. Also, if $n \geq m$ then $y_\alpha|n$ for $\alpha \in Y$ are pairwise distinct. So we can extend f to $g \in \mathscr{P}$ by letting $g(y_\alpha|n) = p(\alpha)(n)$ for $\alpha \in Y$ and $m \leq n < \mathrm{ht}(p)$. It is now easy to check that g works. $\square$

Corollary 7.4. *Forcing via $\mathscr{C}_X$ adjoins a weak $\mathscr{M}$-generic sequence on Q_X.*

Proof. In the generic extension via $\mathscr{C}_X$ we have $|X| \leq 2^\omega$ (recall that X is infinite). Now write $\mathscr{C}_X$ as the iteration $\mathscr{C}_X * \mathscr{C}$ and use the previous lemma. $\square$

Remark. One may ask whether weak $\mathscr{M}$-generic sequences are $\mathscr{M}$-generic. This is not true in general. For example, assume that $\omega_1 \leq |X| \leq 2^\omega$ in $\mathscr{M}$. Then the weak $\mathscr{M}$-generic sequence constructed in Lemma 7.3 in not $\mathscr{M}$-generic. To prove this we argue by contradiction. We can generalize Lemma 7.1 such that we ommit the second clause and then we obtain a complete embedding of some reducts $\mathrm{RO}(Q_X)|p$ into $\mathscr{C}|b \cong \mathscr{C}$ which is not true.

Next we show that weak generic sequences are invariant under finite modifications.

Definition. If $f \in \omega^\omega$ and $t \in \omega^{<\omega}$ let $f \dotdiv t \in \omega^\omega$ be defined as follows:

$$f \dotdiv t(n) = \begin{cases} t(n) & \text{if } n < |t|; \\ f(n) & \text{otherwise.} \end{cases}$$

Lemma 7.5. *Let $\langle f_\alpha : \alpha \in X \rangle \subseteq \omega^\omega$ be a weak $\mathscr{M}$-generic sequence on Q_X and let $\langle t_\alpha : \alpha \in X \rangle \subseteq \omega^{<\omega}$. Then $\langle f_\alpha \dotdiv t_\alpha : \alpha \in X \rangle$ is also a weak $\mathscr{M}$-generic on Q_X.*

Proof. Note that $\langle t_\alpha : \alpha \in X \rangle$ need not be in $\mathscr{M}$. Of course $\langle f_\alpha \dotdiv t_\alpha : \alpha \in X \rangle$ is almost disjoint. To prove weak genericity fix a finite $Y \subseteq X$ and a dense $D \subseteq Q_Y$ from $\mathscr{M}$.

We shall find $s \in D$ coherent with $\langle f_\alpha \dotdiv t_\alpha : \alpha \in Y \rangle$. Let m be large enough such that $\{f_\alpha : \alpha \in Y\}$ is disjoint above m and $|t_\alpha| \leq m$ for $\alpha \in Y$. Let $p, q \in Q_Y$ be such that $\mathrm{dom}(p) = \mathrm{dom}(q) = Y$, $p(\alpha) = f_\alpha|m$ and $q(\alpha) = f_\alpha \div t_\alpha|m$. Then p is coherent with $\langle f_\alpha : \alpha \in Y \rangle$.

Now we use the homogenity argument. If $r \in Q_Y$ and $r \leq p$ write $r = p \cup w$ where w is the part of r above $m = \mathrm{ht}(p) = \mathrm{ht}(q)$. Put $\pi(r) = q \cup w$. It is easy to see that $\pi : Q_Y|p \to Q_Y|q$ is an order isomorphism (in $\mathcal{M}$). Consequently, the set $\pi^{-1}(D)$ is dense below p. Hence, by weak genericity, for some r coherent with $\langle f_\alpha : \alpha \in Y \rangle$ we have $r \leq p$ and $\pi(r) \in D$. Thus $s = \pi(r)$ works. $\square$

Now we show how to obtain an $\mathcal{M}$-generic sequence on Q_X by modifying a given weak $\mathcal{M}$-generic. The modification uses a sequence $\langle t_\alpha : \alpha \in X \rangle \subseteq \omega^{<\omega}$ obtained generically from the following poset T_X which is just another version of Cohen forcing $\mathcal{C}_X$. This kind of argument resembles the one described in [T]: adding a Cohen real over "almost" generic object produces generic.

The elements of T_X are finite functions t such that $\mathrm{dom}(t) \subseteq X$ and $\mathrm{ran}(t) \subseteq \omega^{<\omega} \setminus \{\emptyset\}$. The order on T_X is extension: $t \leq s$ iff $t \supseteq s$. Then T_X is separative and $\mathrm{RO}(T_X) \cong \mathcal{C}_X$ for infinite X. If $\mathcal{G}$ is a generic filter on T_X then $\bigcup \mathcal{G} : X \to \omega^{<\omega}$. Let $t_\alpha = \bigcup \mathcal{G}(\alpha)$ for $\alpha \in X$. The main result is the following.

Proposition 7.6. *Let $\langle f_\alpha : \alpha \in X \rangle$ be a weak $\mathcal{M}$-generic sequence on Q_X and let $\langle t_\alpha : \alpha \in X \rangle$ be a sequence obtained by forcing via T_X over the model $\mathcal{M}[\langle f_\alpha : \alpha \in X \rangle]$. Then $\langle f_\alpha \div t_\alpha : \alpha \in X \rangle$ is an $\mathcal{M}$-generic sequence on Q_X.*

Proof. Fix a dense set $D \subseteq Q_X$ from $\mathcal{M}$. We have to find $p \in D$ coherent with $\langle f_\alpha \div t_\alpha : \alpha \in X \rangle$. We can additionally assume that D is open: if $p \leq q \in D$ then $p \in D$. For $s \in T_X$ write s_α for $s(\alpha)$. Let $p_s \in Q_X$ be defined as follows:

- $\mathrm{dom}(p_s) = \mathrm{dom}(s)$;
- $\mathrm{ht}(p_s) = \max\{|s_\alpha| : \alpha \in \mathrm{dom}(s)\}$;
- $p(\alpha) = (f_\alpha \div s_\alpha)|\mathrm{ht}(p_s)$.

Thus p_s is formed by extending every s_α using f_α up to the maximal length. Let

$$E = \{s \in T_X : p_s \in D \text{ and } p_s \text{ is coherent with } \langle f_\alpha \div s_\alpha : \alpha \in \mathrm{dom}(s) \rangle\}.$$

Note that the function $s \mapsto p_s$ and E are defined in the model $\mathcal{M}[\langle f_\alpha : \alpha \in X \rangle]$.

CLAIM: *E is dense in T_X.*

Before we prove the claim let us show how it ends the proof. E intersects the generic filter on T_X. So let $s \in E$ be such that $s_\alpha = t_\alpha$ for $\alpha \in \mathrm{dom}(s)$. Then $p_s \in D$ and p_s is coherent with $\langle f_\alpha \div t_\alpha : \alpha \in Y \rangle$ and, a fortiori, with $\langle f_\alpha \div t_\alpha : \alpha \in X \rangle$.

PROOF OF THE CLAIM:

Let $r \in T_X$ be given. We have to find $s \in E$ such that $s \leq r$. Wlog $r \neq \emptyset$. Let $Y = \mathrm{dom}(r)$. Choose m large enough such that $m > \max\{|r_\alpha| : \alpha \in Y\}$ and $\{f_\alpha : \alpha \in Y\}$ is disjoint above m.

Define $q \in Q_Y$ by letting $q(\alpha) = f_\alpha \div r_\alpha|m$ for $\alpha \in Y$. Then q is coherent with $\langle f_\alpha \div r_\alpha : \alpha \in Y \rangle$.

By Lemma 7.5 the sequence $\langle f_\alpha \dotdiv r_\alpha : \alpha \in Y\rangle$ *is* $\mathscr{M}$-generic on Q_Y.

Let $D|Y = \{p|Y : p \in D\}$. Then $D|Y \in \mathscr{M}$ and $D|Y$ is a dense subset of Q_Y.

Hence, by genericity below q, there exists $p \in D$ such that $p|Y \leq q$ and $p|Y$ is coherent with $\langle f_\alpha \dotdiv r_\alpha : \alpha \in Y\rangle$.

Choose $n > \mathrm{ht}(p)$ such that $\{f_\alpha : \alpha \in \mathrm{dom}(p)\}$ is disjoint above n.

Let $w \leq p$ be defined as follows. Put $\mathrm{dom}(w) = \mathrm{dom}(p)$. For $\alpha \in Y$ let $w(\alpha) = f_\alpha \dotdiv r_\alpha | n$. For $\alpha \in \mathrm{dom}(p) \setminus Y$ extend $p(\alpha)$'s by arbitrary disjoint sequences (up to the length n) just to ensure that $w \leq p$. Then $w \in D$ because D is open.

Finally, in T_X, let s be the extension of r defined by w. Thus $\mathrm{dom}(s) = \mathrm{dom}(w)$ and $s_\alpha = w(\alpha)$ for $\alpha \in \mathrm{dom}(w) \setminus Y$. Then $p_s = w \in D$ and w is coherent with $\langle f_\alpha \dotdiv s_\alpha : \alpha \in \mathrm{dom}(s)\rangle$.

Thus $s \leq r$ and $s \in E$ as required.

This completes the proof of the CLAIM. □

Corollary 7.7. *If $\mathscr{B}$ is a complete Boolean algebra such that*

$$[\![\textit{there exists a weak } \check{V}\textit{-generic sequence on } Q_X]\!] = \mathbf{1}$$

then $\mathrm{RO}(Q_X)$ *can be completely embedded into the minimal product* $\mathscr{B} \otimes \mathrm{RO}(T_X)$.

Proof. Choose a $\mathscr{B}$-valued name $\langle f_\alpha : \alpha \in X\rangle$ such that

$$[\![\langle f_\alpha : \alpha \in X\rangle \text{ is a weak } \check{V}\text{-generic sequence on } Q_X]\!] = \mathbf{1}.$$

Choose also an $\mathrm{RO}(T_X)$-valued name $\langle t_\alpha : \alpha \in X\rangle$ for generic sequence. By the absolutness of T_X and Product Lemma (cf. [J] Lemma 20.1) the sequence $\langle t_\alpha : \alpha \in X\rangle$ is generic *over* the Boolean model $V^{\mathscr{B}}$. Therefore, by Proposition 7.6 the $\mathscr{B} \otimes \mathrm{RO}(T_X)$-name $\mathscr{G}\langle f_\alpha \dotdiv t_\alpha : \alpha \in X\rangle$ is with value $\mathbf{1}$ $\check{V}$-generic sequence on Q_X. To apply Lemma 7.1 we need to check condition 2. Let $p \in Q_X$ be given. Wlog $p \neq \emptyset$. Let $Y = \mathrm{dom}(p)$. There exist $m > \mathrm{ht}(p)$ and $b \in \mathscr{B}^+$ such that

$$b \Vdash \langle f_\alpha : \alpha \in Y\rangle \text{ is disjoint above } m.$$

Let $q \in Q_Y$ be any condition such that $q \leq p$ and $\mathrm{ht}(q) = m$. Now treat q as element of T_X. Then $\langle b, q\rangle \in (\mathscr{B} \otimes \mathrm{RO}(T_X))^+$ and $\langle b, q\rangle \Vdash p \in \mathscr{G}\langle f_\alpha \dotdiv t_\alpha : \alpha \in X\rangle$. □

Corollary 7.8. $\mathrm{RO}(Q_X)$ *can be completely embedded into* $\mathscr{C}_X$, *if X is infinite.*

Proof. Use Corollaries 7.4 and 7.7. Note that $\mathscr{C}_X \otimes \mathrm{RO}(T_X) \cong \mathscr{C}_X$. □

Let us state now the Shelah-Koppelberg Theorem.

Theorem 7.9. *For every $\kappa \geq \omega_2$ there exists a complete subalgebra of $\mathscr{C}_\kappa$, with uniform density κ, which is not a category algebra.*

Proof. From Proposition 6.7 and Corollary 7.8. □

Notice also the following.

Corollary 7.10. $\mathrm{RO}(Q_{\omega_1}) \cong \mathscr{C}_{\omega_1}$.

Proof. $\mathrm{RO}(Q_{\omega_1})$ is a complete subalgebra of C_{ω_1} and has uniform density ω_1. Now apply Theorem 5.1. $\square$

8 Another construction of embedding

In this section we give an alternative proof that $\mathrm{RO}(Q_X)$ can be completely embedded into $\mathscr{C}_X$. Still, we assume that X is infinite and let $\kappa = |X|$. In fact, without any forcing tools we construct a very concrete neat cover function. This can be done by more careful examination of Corollary 7.7. Such a function was also constructed (independently) by J. Zapletal in [Z].

Let $\mathscr{P}_1$ be the standard poset for adding κ. Cohen reals in ω^ω. The elements of $\mathscr{P}_1$ are finite functions s such that $\mathrm{dom}(s) \subseteq \kappa$ and $\mathrm{ran}(s) \subseteq \omega^{<\omega}\setminus\{0\}$. If $s_1, s_2 \in \mathscr{P}_1$ let $s_1 \leq s_2$ iff $\mathrm{dom}(s_1) \supseteq \mathrm{dom}(s_2)$ and $s_1(\alpha) \supseteq s_2(\alpha)$ for $\alpha \in \mathrm{dom}(s_2)$.

Let $\mathscr{P}_2$ be the poset for adding the function $F : \omega^{<\omega} \to \omega$ described in the proof of Lemma 7.3 as $\mathscr{P}$.

Finally, let $\mathscr{P}_3 = T_\kappa$. Consider the product $\mathscr{P}_1 \times \mathscr{P}_2 \times \mathscr{P}_3$ with coordinate-wise ordering. It is separative and $\mathrm{RO}(\mathscr{P}_1 \times \mathscr{P}_2 \times \mathscr{P}_3) \cong \mathscr{C}_\kappa$. We remark that $\mathscr{P}_1$ is used to increase the cardinality of continuum to (at least) κ and to provide canonical names for a family of distinct functions like in Lemma 7.3. Next, $\mathscr{P}_2$ is an application of Lemma 7.3 to obtain a weak generic sequence. Finally $\mathscr{P}_3$ is the "correction" as in Proposition 7.6. However, we dont need any of those facts here.

First we define the domain of our neat cover function. Let D be the set consisting of all triples $(s, f, t) \in \mathscr{P}_1 \times \mathscr{P}_2 \times \mathscr{P}_3$ such that:

1. $\mathrm{dom}(s) = \mathrm{dom}(t) \neq \emptyset$;
2. there is $n \geq 1$ such that $s(\alpha) \in \omega^n$ and $|t(\alpha)| \leq n$ for all $\alpha \in \mathrm{dom}(s)$;
3. $s(\alpha) : \alpha \in \mathrm{dom}(s)$ are pairwise distinct;
4. $\mathrm{dom}(f) = \{s(\alpha)|m : \alpha \in \mathrm{dom}(s) \text{ and } m < n\}$.

Lemma 8.1. *D is a dense subset of $\mathscr{P}_1 \times \mathscr{P}_2 \times \mathscr{P}_3$.*

Proof. Exercise. $\square$

If follows that D is also separative and $\mathrm{RO}(D) \cong \mathscr{C}_\kappa$. We now define a neat cover function $\varphi : D \to Q_\kappa$. For $(s, f, t) \in D$ let $p = \varphi(s, f, t)$ be the element of Q_κ defined as follows:

- $\mathrm{dom}(p) = \mathrm{dom}(s) = \mathrm{dom}(t)$;
- $\mathrm{ht}(p) = n$ where $n \geq 1$ is from condition (2) defining D;
- for $\alpha \in \mathrm{dom}(p)$ let

$$p(\alpha)(m) = \begin{cases} t(\alpha)(m) & \text{if} \quad m < |t(\alpha)|; \\ f(s(\alpha)|m) & \text{if} \quad |t(\alpha)| \leq m < n. \end{cases}$$

Lemma 8.2. *$\varphi : D \to Q_\kappa$ is neat cover.*

Proof. We check conditions from Lemma 2.1.

(1) $\operatorname{ran}(\varphi) = Q_\kappa \setminus \{\emptyset\}$.

Let $\emptyset \neq p \in Q_\kappa$ and $n = \operatorname{ht}(p)$. Pick *arbitrary* distinct sequences $\{s(\alpha) : \alpha \in \operatorname{dom}(p)\} \subseteq \omega^n$ and $f \in \mathscr{P}_2$ such that (4) from definition of D holds. As $Q_\kappa \subseteq T_\kappa$ we have $(s, f, p) \in D$ and $\varphi(s, f, p) = p$.

(2) φ is order preserving.

Assume that $(s_1, f_1, t_1) \leq (s_2, f_2, t_2)$. Let $p = \varphi(s_1, f_1, t_1)$ and $q = \varphi(s_2, f_2, t_2)$. Then for $\alpha \in \operatorname{dom}(s_2)$ we have $s_1(\alpha) \supseteq s_2(\alpha)$ and $t_1(\alpha) = t_2(\alpha)$. Therefore $p(\alpha) \supseteq q(\alpha)$ for $\alpha \in \operatorname{dom}(s_2) = \operatorname{dom}(q)$. Also $s_2(\alpha) : \alpha \in \operatorname{dom}(q)$ are distinct. Hence $s_1(\alpha)|m : \alpha \in \operatorname{dom}(q)$ are also distinct if $\operatorname{ht}(q) \leq m < \operatorname{ht}(p)$. But $f_1 \in \mathscr{P}_2$ and so $\{p(\alpha) : \alpha \in \operatorname{dom}(q)\}$ is disjoint above $\operatorname{ht}(q)$. Thus $p \leq q$.

(3) If $p \leq \varphi(s_2, f_2, t_2) = q$ then $p = \varphi(s_1, f_1, t_1)$ for some $(s_1, f_1, t_1) \leq (s_2, f_2, t_2)$.

Let $Y = \operatorname{dom}(q)$. We have $(s_2, f_2, t_2) \in D$. So $s_2(\alpha) : \alpha \in Y$ are pairwise distinct. Also $\{p(\alpha) : \alpha \in Y\}$ is disjoint above $\operatorname{ht}(q)$. So it is possible to extend inductively every $s_2(\alpha)$ to $s_1(\alpha) \in \omega^{\operatorname{ht}(p)}$ and simultaneously define $f_1 \in \mathscr{P}_2$ such that $p(\alpha)(m) = f_1(s_1(\alpha)|m)$ for $\operatorname{ht}(q) \leq m < \operatorname{ht}(p)$. The same argument was used in the end of the proof of Lemma 7.3. For $\alpha \in \operatorname{dom}(p) \setminus Y$ put $t_1(\alpha) = p(\alpha)$ and for $s_1(\alpha)$ pick new disjoint sequences from $\omega^{\operatorname{ht}(p)}$. Finally define f_1 arbitrary on restrictions to have $(s_1, f_1, t_1) \in D$. Then $\varphi(s_1, f_1, t_1) = p$. $\square$

We end the paper by the following problem.

Problem. *Characterize complete subalgebras of Cohen algebra $\mathscr{C}_\kappa$.*

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