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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 41 (2000), No. 2, 7--11

Persistent URL: http://dml.cz/dmlcz/702064

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Received 11. March 2000

B. R. Hunt recently proved that the set S of continuous functions $f \in C[0, 1]$ which have a finite derivative at a point $x \in (0\ 1)$ is Haar null in Christensen's sense. Let μ be the Wiener measure on $C[0, 1]$. We show that a natural very slight modification of the well-known simple proof of A. Dvoretzky, P. Erdös, and S. Kakutani that $\mu(S) = 0$ gives that $\mu(S + f) = 0$ for each $f \in C[0, 1]$, which gives Hunt's result. A related conjecture concerning Jarník points is formulated.

It was shown by S. Banach in [Ba] and S. Mazurkiewicz in [M] that, in topological sense, most continuous functions $f \in C[0, 1]$ are nondifferentiable at every point of $(0, 1)$. More precisely, if we put

$$S := \{f \in C[0, 1];\ f \text{ has a derivative } f'(x) \in \mathbb{R} \text{ at a point } x \in (0, 1)\},$$

then S is a first category set in $C[0, 1]$. It is well-known that S is also null in a measure sense, namely that $\mu(S) = 0$, where μ is the Wiener measure on $C[0, 1]$. Hunt recently proved that S is null in another measure sense. He proved (using a different terminology) that S is Haar null in the Christensen sense.

We say here that a subset H of a separable Banach space X is a *Haar null set* if there exists a complete Borel probability measure μ on X and a universally measurable set $U \subset X$, such that $H \subset U$ and

$$(*) \qquad \mu(U + f) = 0 \quad \text{for every } f \in X.$$

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1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 26A27; Secondary 46G12, 60J65.

KEY WORDS and PHRASES

Nondifferentiable functions, Haar null sets, Wiener measure, Jarník points.

Our investigation was supported by GAČR 201/00/0767, and CEZ J13/98113200007.

We comment other variants of this notion that appear in the literature in our concluding remark.

In this note we present our observation that Hunt's result on Haar nullness of S can be easily derived from known facts concerning the Wiener measure. In fact, a natural slight modification of the Dvoretzky, Erdös, and Kakutani proof ([DEK]) which shows in particular that $\mu(S) = 0$, where μ is the Wiener measure on $C[0, 1]$, gives (*) with $X = C[0, 1]$ and a Borel U containing S.

We came to this observation when we tried to prove the following conjecture of the second named author.

Let us first recall (cf. [GH], [AHP], [MZ]) that $x \in (0, 1)$ is a *Jarník point* of f if

$$\text{ap} - \lim_{t \to x} \left| \frac{f(t) - f(x)}{t - x} \right| = \infty ,$$

where ap $-$ lim stands for the approximate limit.

Conjecture. *Let μ be the Wiener measure on $C[0, 1]$ and let N be the set of those $f \in C[0, 1]$ with have a non-Jarník point in $(0, 1)$. Then $\mu(N + f) = 0$ for every $f \in C[0, 1]$. In particular, since N is universally measurable, N is Haar null (in the Christensen sense).*

It was proved by S. M. Berman [Be] that $\mu(N) = 0$ (cf. also [GH], [AHP]; concerning Jarník points of typical $f \in C[0, 1]$ in the category sense, see [MZ]). We do not know whether the above conjecture is true. J. Kolář [Ko] recently improved Hunt's result in several directions. In particular, he proved that L_{ap} is Haar null, where

$$L_{\text{ap}} := \left\{ f \in C[0, 1];\ \text{ap} - \limsup_{t \to x+} \left| \frac{f(t) - f(x)}{t - x} \right| < \infty \text{ at a point } x \in [0, 1) \right\}.$$

Notice that the statement of our conjecture is stronger than the mentioned result of Kolář.

Now we are going to formulate the result and its proof that is just a very small modification of the well-known simple proof of Dvoretzky, Erdös, and Kakutani that almost all trajectories of the Wiener process are not right Lipschitz in any point.

Recall that f is *Lipschitz from the right* at x if

$$\limsup_{y \to x+} \left| \frac{f(y) - f(x)}{y - x} \right| < \infty .$$

Theorem. *Let μ be the Wiener measure on $C[0, 1]$ and L be the set of those $f \in C[0, 1]$ which are Lipschitz from the right at a point $x \in [0, 1)$. Then $\mu(L + f) = 0$ for each $f \in C[0, 1]$.*

Let us remark that in particular L, and also S as a subset of L, is Haar null (see the definition and remark above). This result is however covered by the mentioned Kolář's result.

Let us give, or recall, some notation before going to the proof of Theorem. We use $e_t : C[0,1] \to \mathbb{R}$ to denote the evaluation mapping defined by $e_t(f) = f(t)$. For $\triangle > 0$, we write $\nu_\triangle$ to denote the Gaussian probability measure on $\mathbb{R}$ with the density (with respect to the Lebesgue measure λ) equal to

$$\frac{d\nu_\triangle}{d\lambda}(x) = \gamma_\triangle(x) = \frac{1}{\sqrt{2\pi\triangle}} \mathrm{e}^{-\frac{x^2}{2\triangle}}.$$

Wiener measure μ on $C[0,1]$ is the completion of the unique Borel probability measure on $C[0,1]$ such that

(1) $\mu(\{f \in C[0,1];\ f(0) = 0\}) = 1$ and

(2) for every choice of $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \dots < t_n \leq 1$, the image of μ by the mapping $(e_{t_1} - e_{t_0}, \dots, e_{t_n} - e_{t_{n-1}}) : C[0,1] \to \mathbb{R}^n$ is the product measure $\nu_{t_1 - t_0} \otimes \dots \otimes \nu_{t_n - t_{n-1}}$ in $\mathbb{R}^n$.

Proof of Theorem. Let $f \in C[0,1]$ be arbitrary. We are going to show that

$$\mu(f + L) = \mu(\{g \in C[0,1];\ g - f \in L\}) = 0.$$

We do first an obvious observation. Let $x \in [0,1)$ and $h \in L$ be such that h is Lipschitz from the right at x. Then there is a $k \in \mathbb{N}$ and a $\delta > 0$ such that

$$|h(y) - h(x)| \leq k|y - x| \text{ for } y \in (x, x + \delta).$$

There is clearly an $m \in \mathbb{N}$ such that, for every $n \geq m$ and for the minimal $i \in \{0, \dots, n\}$ with $\frac{i}{n} \geq x$, we have that

$$\left\{\frac{i}{n}, \frac{i+1}{n}, \frac{i+2}{n}, \frac{i+3}{n}\right\} \subset [x, x + \delta).$$

It is easy to check that

$$\left|h\left(\frac{j+1}{n}\right) - h\left(\frac{j}{n}\right)\right| \leq 7\frac{k}{n} \quad \text{for } j \in \{i, i+1, i+2\}.$$

Therefore we see that

$$L \subset \bigcup_{k \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} \bigcup_{i=0}^{n} \bigcap_{j=0}^{i+2} \left\{h \in C[0,1];\ \left|h\left(\frac{j+1}{n}\right) - h\left(\frac{j}{n}\right)\right| \leq 7\frac{k}{n}\right\}, \text{ and so}$$

(**)
$$f + L \subset \bigcup_{k \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} \bigcup_{i=0}^{n} M_{k,n,i},$$

where

$$M_{k,n,i} = \bigcap_{j=i}^{i+2} \Big\{ g \in C[0,1];$$
$$g\left(\frac{j+1}{n}\right) - g\left(\frac{j}{n}\right) \in \left[f\left(\frac{j+1}{n}\right) - f\left(\frac{j}{n}\right) - \frac{7k}{n}, f\left(\frac{j+1}{n}\right) - f\left(\frac{j}{n}\right) + \frac{7k}{n} \right] \Big\}.$$

Since

$$\left(e_{\frac{i+1}{n}} - e_{\frac{i}{n}}, e_{\frac{i+2}{n}} - e_{\frac{i+1}{n}}, e_{\frac{i+3}{n}} - e_{\frac{i+2}{n}}\right)(\mu) = \nu_{\frac{1}{n}} \otimes \nu_{\frac{1}{n}} \otimes \nu_{\frac{1}{n}},$$

we obtain that

$$\mu(M_{k,n,i}) = \nu_{\frac{1}{n}}(I_i)\, \nu_{\frac{1}{n}}(I_{i+1})\, \nu_{\frac{1}{n}}(I_{i+2}),$$

where each I_j is an interval of length $\lambda(I_j) = \frac{14k}{n}$.

Since

$$\nu_{\frac{1}{n}}(I_j) \le \lambda(I_j) \frac{1}{\sqrt{2\pi \frac{1}{n}}} \quad \text{for } j \in \{i, i+1, i+2\},$$

we obtain

$$\mu\left(\bigcup_{i=0}^{n} M_{k,n,i}\right) \le (n+1)\left(\frac{1}{\sqrt{2\pi \frac{1}{n}}} \frac{14k}{n}\right)^3 = \left(\frac{14k}{\sqrt{2\pi}}\right)^3 \left(\frac{n+1}{n^{\frac{3}{2}}}\right).$$

Since the sequence of these numbers tends to zero with n tending to infinity, $\mu(f + L) = 0$ follows by (**) above.

Remark on the notion of Haar null sets. Our definition above is a bit weaker than those appearing in [Ch] or [BL] within the separable Banach spaces. The definition in [Ch] assumes that H is universally measurable itself and the definition in [BL] assumes even that H is Borel. We are pointing out our weak one because it seems sufficient to demonstrate the smallness of the sets S and L in our theorem above (and in the remark following it).

Nevertheless, we should remark that in fact we find in both cases a Borel set U containing S, or L, that fulfils (*) with μ being the Wiener measure.

Moreover, it is well-known that the set S is an analytic (non-Borel) subset of $C[0,1]$ (cf. [Ke, Theorem 33.15]). So S is universally measurable and being Haar null in our above sense gives also that it is Haar null in the original Christensen's sense (cf. [Ch]).

Our set L is Borel, in fact it is an F_σ subset of $C[0,1]$. One may check it by noticing that $L = \bigcup_{n \in \mathbb{N}} E_n$ with the closed sets E_n taken from [O, Chpt. 11, p. 45]. This shows that L being Haar null by the above definition is also Haar null in both the sense of [BL] and of [Ch].

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