Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 40 (1999), No. 2, 27--41

Persistent URL: http://dml.cz/dmlcz/702055

1999 ACTA UNIVERSITATIS CAROLINAE – MATHEMATICA ET PHYSICA VOL. 40, NO. 2

Exhaustive Zero-Convergence Structures on Boolean Algebras

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Received 11. March 1999

The aim of the paper is to describe the necessary and sufficient conditions for a Boolean algebra to admit the largest possible sequential convergence structure. We present examples of complete algebras, known from construction of various generic extensions of set theory, carrying such convergence structures.

1. Introduction

In this section we review some basic notions and facts concerning sequential convergence structures and continuity of submeasures on a Boolean algebra B. The motivation for the research described in this paper comes from [Ja] and [Ja1], where it is shown that the maximal possible convergence structure is attained for $(\omega, 2)$-distributive Boolean algebras. We give the necessary and sufficient conditions for a Boolean algebra to admit the largest possible sequential convergence structure. Furthermore, we prove that for any algebra B and any sequential convergence structure s on the algebra, the join $s \vee os$ in the semilattice of all convergence structures on B exists. os is the classical order convergence structure on B. Jakubík in [Ja] proved this for the convergence structure induced by D (see 2.2 below). We would like to thank Zbigniew Lipecki for many valuable comments.

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Supported in part by a GA ČR grant 201/97/0216 (Balcar) and by an NSERC grant OGP0025112 (Franek). Franek and Hruška both acknowledge the hospitality of the Center for Theoretical Study in Prague, Balcar acknowledges the hospitality of McMaster University. *1981 Mathematics Subject Classifica ion.* Primary 28A60, 06E10; Secondary 09E55, 54A20, 54A25. *Key words and phrases:* Boolean a gebra, convergence structure, sequential topology, exhaustive submeasures.

Throughout this paper we consider only non-degenerate Boolean algebras with $\mathbf{0} \neq \mathbf{1}$. For a Boolean algebra B, we view infinite sequences of elements of B as elements of B^ω, the product of ω copies of B. B^ω is also a Boolean algebra.

The fundamental reference concerning Boolean algebras is [K] and concerning convergence structures on Boolean algebras Vladimirov's book [VI]. Some useful information pertaining the topic of convergence structures and submeasures on rings and fields of subsets can be found in [De].

Let us first introduce the largest possible zero convergence structure on B, $\mathscr{Z}(B)$. $\mathscr{Z}(B) = \{f : \omega \to B : (\forall I \in [\omega]^\omega) \bigwedge\{f(n) : n \in I\} = \mathbf{0}\}$. Clearly, $\emptyset \neq \mathscr{Z}(B) \subseteq B^\omega$.

When clear from the context, we drop the reference to B and use just $\mathscr{Z}$, including other structures defined on Boolean algebras that will be introduced later.

We shall denote by $\varphi \in \omega^\omega\uparrow$ the fact that φ is a strictly increasing sequence of non-negative integers.

1.1 Definition. Let B be a Boolean algebra and let $\mathscr{I}$ be an ideal on B^ω. $\mathscr{I}$ is said to be a *zero-convergence structure on B* if

(i) $\mathscr{I} \subseteq \mathscr{Z}$,

(ii) $\mathscr{I}$ is closed under subsequences, i.e. whenever $f \in \mathscr{I}$ and $\varphi \in \omega^\omega\uparrow$, then $f \circ \varphi \in \mathscr{I}$.

Note that $\mathscr{Z}$ need not be a zero-convergence structure on B. Consider a Cantor algebra $\mathscr{A}$, i.e. $\mathscr{A} \approx Clop(2^\omega)$, the algebra of clopen subsets of the Cantor space 2^ω. Equivalently, $\mathscr{A}$ is a free algebra with countably many independent generators, say $\langle x_n : n \in \omega\rangle$. Then f defined by $f(n) = x_n$ belongs to $\mathscr{Z}(\mathscr{A})$, and so does $-f = \langle -x_n : n \in \omega\rangle$. Since $f \vee -f = \mathbf{1}_{\mathscr{A}^\omega}$, $\mathscr{Z}(\mathscr{A})$ cannot be an ideal.

Therefore, the largest possible zero-convergence structure with respect to $\subseteq$ need not be a zero-convergence structure at all, nevertheless the maximality principle is applicable, hence each zero-convergence structure on B can be extended to a maximal one.

Conditions under which $\mathscr{Z}(B)$ itself is a zero-convergence structure are discussed in section 3.

1.2 Definition. Let B be a Boolean algebra and let $A \subseteq B^\omega$. *Urysohn closure* of $A, \mathscr{U}(A)$, is a subset of B^ω with the property that every subsequence of a sequence from $\mathscr{U}(A)$ has a subsequence that belongs to A, i.e. $\mathscr{U}(A) = \{f \in B^\omega : (\forall\varphi \in \omega^\omega\uparrow)(\exists\psi \in \omega^\omega\uparrow)(f \circ \varphi \circ \psi) \in A\}$.

The following are easy observations.

1.3 Fact. (i) *for any $A \subseteq B^\omega$, $\mathscr{U}(\mathscr{U}(A)) = \mathscr{U}(A)$,*

(ii) *$\mathscr{Z}$ is Urysohn closed,*

(iii) *if $\mathscr{I}$ is a zero-convergence structure, then $\mathscr{U}(\mathscr{I})$ is a zero-convergence structure, too.*

Let $s \subseteq B^\omega \times B$. If $(\langle x_n : n \in \omega\rangle, x) \in s$, we write $x_n \underset{s}{\to} x$. x is said to be an *s-limit of the sequence* $\langle x_n : n \in \omega\rangle$. For $a \in B$, k_a donotes the constant sequence $k_a(n) = a$. Thus k is a natural embedding of B into B^ω.

If $\mathcal{I}$ is a zero-convergence structure on B, we consider the elements of $\mathcal{I}$ as sequences converging to $\mathbf{0}$. We can extend this to a notion of convergence $s(\mathcal{I})$ of sequences on B by defining $x_n \underset{s(\mathcal{I})}{\rightarrow} x$ whenever $\langle x_n \triangle x : n \in \omega \rangle \in \mathcal{I}$, where $\triangle$ denotes the Boolean operation of *symmetric difference*. It is easy to verify that the following holds.

1.4 Fact. (i) *every sequence has at most one limit, i.e. if* $x_n \underset{s(\mathcal{I})}{\rightarrow} x$ *and* $x_n \underset{s(\mathcal{I})}{\rightarrow} y$, *then* $x = y$,

(ii) *if* $x \in B$, *then the constant sequence* $\langle x : n \in \omega \rangle$ *has* x *as its limit,*

(iii) *if* $x_n \underset{s(\mathcal{I})}{\rightarrow} x$ *and* $\langle y_n : n \in \omega \rangle$ *is a subsequence of* $\langle x_n : n \in \omega \rangle$, *then* $y_n \underset{s(\mathcal{I})}{\rightarrow} x$,

(iv) *if* $x_n \leq y_n \leq z_n$ *for every* n *and* $x_n \underset{s(\mathcal{I})}{\rightarrow} x$ *and* $z_n \underset{s(\mathcal{I})}{\rightarrow} x$, *then* $y_n \underset{s(\mathcal{I})}{\rightarrow} x$,

(v) *the convergence respects Boolean operations, i.e. if* $x_n \underset{s(\mathcal{I})}{\rightarrow} x$ *and* $y_n \underset{s(\mathcal{I})}{\rightarrow} y$, *then* $x_n \vee y_n \underset{s(\mathcal{I})}{\rightarrow} x \vee y$ *and* $-x_n \underset{s(\mathcal{I})}{\rightarrow} -x$.

The notions of zero convergence and convergence are really identical in the sense that a convergence structure $s(\mathcal{I})$ induced by a zero-convergence structure $\mathcal{I}$ is a convergence structure on B, i.e. a structure satisfying 1.4 (i)–(v), while for a convergence structure s on B, $s_0 = \{f \in B^\omega : f(n) \underset{s}{\rightarrow} \mathbf{0}\}$ is $s_0 \subseteq \mathcal{X}$ and a zero-convergence structure on B.

Let us recall some of the basic notions concerning sequential topologies.

1.5 Definition. Let (X, τ) be a topological space. X is said to be

(i) *sequential* if any $A \subseteq X$ is closed whenever it contains all limits of τ-convergent sequences of elements of A,

(ii) *Fréchet* if for any $A \subseteq X$,

$$cl_r(A) = \{x \in X : (\exists \langle x_n : n \in \omega \rangle \subseteq A)\, x_n \underset{r}{\rightarrow} x\}.$$

It is clear that every Fréchet space is sequential.

A convergence structure s on a Boolen algebra B gives rise to a sequential topology on B in the following way: consider all topologies τ on B so that whenever $x_n \underset{s}{\rightarrow} x$, then $x_n \underset{\tau}{\rightarrow} x$. There is a largest topology with respect to inclusion among all such topologies, and we denote it by $\tau(s)$ and call it *sequential topology determined by s*.

Alternatively, the topology $\tau(s)$ can be described through the closure operation: for $A \subseteq B$, let $u(A) = \{x : x$ *is the s-limit of a sequence* $\langle x_n \rangle$ *of elements of* $A\}$. Then $cl_{\tau(c)}(A) = \bigcup_{\alpha < \omega_1} u^{(\alpha)}(A)$, where $u^{(\alpha+1)}(A) = u(u^{(\alpha)}(A))$ and $u^{(\alpha)}(A) = \bigcup\{u^{(\beta)}(A) : \beta < \alpha\}$ for a limit α.

It follows from 1.4 (ii) that every singleton is a closed set, i.e. $\tau(s)$ is a T_1 topology. Moreover, $(B, \tau(s))$ is a sequential topological space, and it is Fréchet if and only if $cl_{\tau(s)}(A) = u(A)$ for every $A \subseteq B$.

1.6 Fact. *A sequence* $\langle x_n \rangle$ *converges to* x *in the topology* $\tau(s)$, $x_n \underset{\tau(s)}{\rightarrow} x$, *if and only if any subsequence of* $\langle x_n \rangle$ *has a subsequence that converges to* x *in* s.

Le $\mathscr{I}$ be a zero-convergence structure on a Boolean algebra B. In the way described above, the convergence structure $s(\mathscr{I})$ determines a sequential topology, which we denote by $\tau(\mathscr{I})$. It follows that the Urysohn closure of $\mathscr{I}$, $U(\mathscr{I})$, is the set of all sequences of elements of B that converge to $\mathbf{0}$ in the topology $\tau(\mathscr{I})$. Moreover, if $\mathscr{I}$ is Urysohn closed, then $x_n \underset{s(\mathscr{I})}{\rightarrow} x$ iff $x_n \underset{\tau(\mathscr{I})}{\rightarrow} x$.

The following characterisation of continuity of mappings is well known.

1.7 Fact. *Let $\tau(s)$ be a topology on B determined by a convergence structure s and let (Y, τ) be an arbitrary topological space. Then a mapping $f : B \to Y$ is continuous if and only if $x_n \underset{s}{\rightarrow} x$ implies $f(x_n) \underset{\tau}{\rightarrow} f(x)$.*

Let B be a Boolean algebra. A *submeasure* on B is a function $\mu : B \to \mathbf{R}^+$ with the properties

(i) $\mu(\mathbf{0}) = 0$,
(ii) $\mu(a) \leq \mu(b)$ whenever $a \leq b$ (monotone),
(iii) $\mu(a \vee b) \leq \mu(a) + \beta(b)$ (subadditive).

A submeasure μ on B is

(iv) *exhaustive* if $\lim \mu(a_n) = 0$ for every sequence $\{a_n : n \in \omega\}$ of disjoint elements,
(v) *strictly positive* if $\mu(a) = 0$ only if $a = \mathbf{0}$,
(vi) a (finitely additive) *measure* if for any disjoint a and b, $\mu(a \vee b) = \mu(a) + \mu(b)$.

Any measure is a (uniformly) exhaustive submeasure, since it has a finite norm; for more on submeasures see [Fr].

If B is a Boolen algebra, B^+ denotes the set of all non-zero elements of B, i.e. $B^+ = B - \{\mathbf{0}\}$.

1.8 Fact. (i) *Let $\mathscr{I}$ be a zero-convergence structure on B. Then for any submeasure μ on B, μ is continuous in $\tau(\mathscr{I})$ if and only if $(\forall f \in \mathscr{I})\ \mu(f(n)) \to 0$.*

(ii) *Let S be a non-empty set of submeasures on B such that for any $a \in B^+$ there is some $\mu \in S$ with $\mu(a) > 0$, then $\{f \in B^\omega : (\forall \mu \in S) \lim \mu(f(n)) = 0\}$ is a Urysohn closed zero-convergence structure.*

1.9 Example. Let $B = \mathscr{P}(X)$ for an infinite set X. A sequence $\langle X_n : n \in \omega \rangle$, $X_n \subseteq X$, belongs to $\mathscr{X}$ if and only if $\{X_n : n \in \omega\}$ is point-finite family of sets. Moreover, $\mathscr{X}$ is a zero-convergence structure. For this example let s denote the convergence structure induced by $\mathscr{X}$. When we identify $\mathscr{P}(X)$ with 2^X via characteristic functions, then the convergence in the topology $\tau(s)$ is exactly the pointwise convergence of sequences on 2^X. It is well known that the corresponding sequential topology $\tau(s)$ on 2^X is a product topology if and only if X is countable, see [Ba]. For an uncountable X, the sequential space $(2^X, \tau(s))$ is a Hausdorff, but not a regular, topological space, see [Gl] or [BGJ]. Moreover, the topology $\tau(s)$ is stronger than the usual product topology τ on 2^X. If we consider the spaces of continuous real-valued functions on 2^X with respect to those two topologies, it is shown in [BH] that $\mathscr{C}(2^X, \tau)) \not\subseteq \mathscr{C}(2^X, \tau(s))$ iff size of X is at least as large as the first submeasurable cardinal.

2. Exhaustive zero-convergence

In this section we discuss a hierarchy of (zero) convergence structures on B. We define exhaustive convergence structures motivated by exhaustive submeasures and their continuity. We introduce and characterize a zero-convergence structure $\mathscr{E}$ that is an intersection of all maximal zero-convergence structures.

In the following we will discuss some examples of zero-convergence structures reaching examples when the whole of $\mathscr{Z}$ is a zero-convergence.

2.1 Example. For $f \in B^\omega$, the set $\{n : f(n) \neq \mathbf{0}\}$ is a *support of f*. We shall call f a *finite* element of B^ω if its support is finite. Set

$$Fin(B) = \{f \in B^\omega : f \text{ has a finite support}\}.$$

Fin is the least Urysohn closed zero-convergence and the topology $\tau(Fin)$ it determines is discrete.

Although the ideal *Fin* is not very interesting from the convergence point of view, it becomes more interesting in the context of quotiens algebras. For any B, the quotient algebra B^ω/Fin is σ-closed, i.e. any descending sequence of non-zero elements has a non-zero lower bound. If B has a dense subset of size $\leq 2^\omega$, then B has a base tree (not necessarily homogeneous in height). For the basic case when $B = \{\mathbf{0}, \mathbf{1}\}$ and hence $B^\omega/Fin = \mathscr{P}(\omega)/fin$, see [BPS]. Let $\mathscr{A}$ be the Cantor algebra. Under the CH, the algebras of $\mathscr{A}^\omega/Fin$ and $\mathscr{P}(\omega)/fin$ are the same. Recently Dow [Do] solved the long-standing problem and showed that consistently the completions of those two algebras may be different. Moreover, the height of $\mathscr{A}^\omega/Fin$ can be smaller than that of $\mathscr{P}(\omega)/fin$.

2.2 Example. Let k be a positive integer, $d \in B^\omega$ is called a *k-disjoint sequence*, if for any $X \subset \omega$ of size k, $\bigwedge\{d(n) : n \in X\} = \mathbf{0}$. We use the term *disjoint sequence* for a 2-disjoint sequence. Set

$$D(B) = \{f \in B^\omega : (\exists m \in \omega)\, (\forall X \in [\omega]^m) \bigwedge \{f(i) : i \in X\} = \mathbf{0}\}.$$

It is clear that $Fin \subseteq D$.

2.3 Proposition. *D is a zero-convergence structure generated by all disjoint sequences, i.e. for any $f \in D$ there are disjoint sequences $d_1, \ldots, d_k$ so that $f \leq d_1 \vee \ldots \vee d_k$.*

Proof. When $d_1, \ldots, d_k$ are disjoint sequences, then for any $X \subseteq \omega$, $|X| = k + 1$, using the usual distributivity and the pigeon hole principle, $\bigwedge\limits_{i \in X} d_1(i) \vee \ldots \vee d_k(i) = \mathbf{0}$.

We shall argue the opposite direction using induction. Let $f \in D$ be an m-disjoint sequence. If $m = 2$, f is disjoint, and so we assume that $m > 2$.

Put $d(n) = f(n) - \bigvee_{i<n} f(i)$ for every $n \in \omega$. Then d is a disjoint sequence. We show that $g = f - d$ is $(m-1)$-disjoint. For every $n \in \omega$ we have $g(n) = f(n) \wedge \bigvee_{i<n} f(i)$, in particular, $g(0) = \mathbf{0}$. Let us check that for arbitrary $x_1 < x_2 < \ldots < x_{m-1}$ we get $\bigwedge_{j=1}^{m-1} g(x_j) = \mathbf{0}$.

In case of $x_1 = 0$ we are done, so assume that $x_1 > 0$. We have

$$\bigwedge_{j=1}^{m-1} g(x_j) = \bigwedge_{j=1}^{m-1}\left(f(x_j) \wedge \bigvee_{i<x_j} f(i)\right) = \bigvee_{i<x_1}\left(f(i) \wedge \bigwedge_{j=1}^{m-1} f(x_j)\right).$$

Since f is m-disjoint each member of the latter join is $\mathbf{0}$, hence g is a $(m-1)$-disjoint sequence. □

2.4 Definition. Let $\mathscr{I}$ be a zero-convergence structure. $\mathscr{I}$ is said to be *exhaustive* if $D \subseteq \mathscr{I}$. The induced convergence structure $s(\mathscr{I})$ and the topology $\tau(\mathscr{I})$ it determines are said to be *exhaustive* if $\mathscr{I}$ is.

The following fact gives the motivation and justification of the term "exhaustive zero-convergence" introduced in 2.4. It follows immediately from the definition of exhaustivity and Fact 1.8 (i).

2.5 Fact. *For any submeasure μ on B, μ is exhaustive if and only if it is continuous in the $\tau(D)$ topology.*

We can ask about the description of the largest zero-convergence structure in which all exhaustive submeasures are continuous.

2.6 Example. Put

$$L(B) = \{f \in B^{\omega} : (\forall X \subseteq \omega, \textit{ infinite})(\exists Y \subseteq X, \textit{ finite}) \bigwedge\{f(i) : i \in Y\} = \mathbf{0}\}.$$

The following theorem is a modification of the result of R. Frič, [Fč], who proved the theorem for measures. This is one of the situations when the global properties of exhaustive submeasures and measures on Boolean algebras are the same.

2.7 Theorem. *L is a Uryshon closed zero-convergence structure. Moreover, $L = \{f \in B^{\omega} :$ for every exhaustive submeasure μ on B, $\mu(f(n)) \to 0\}$.*

In the proof of the theorem in addition to Frič's methods we are going to use the following folklore concerning exhaustive submeasures.

2.8 Proposition. *Let μ be a submeasure on B. Then μ is exhaustive if and only if for any $\langle x_n : n \in \omega\rangle \in B^{\omega}$ and any $\varepsilon > 0$, there is a $k \in \omega$ such that for every $p \geq k$, $\mu(\bigvee_{i \leq p} x_i - \bigvee_{i \leq k} x_i) < \varepsilon$.*

Proof. Assume that μ is exhaustive and further assume that there are $\langle x_n : n \in \omega\rangle$ and $\varepsilon > 0$ violating the proposition. Then we can pick an increasing sequence of integers $\langle k_i : i \in \omega \rangle$ such that for each i, $\mu(\bigvee_{i \le k_{i+1}} x_i - \bigvee_{i \le k_i} x_i) \ge \varepsilon$. If we set $y_i = \bigvee_{i \le k_{i+1}} x_i - \bigvee_{i \le k_i} x_i$, then $\langle y_i \rangle$ is a disjoint sequence and $\lim \mu(y_i) \neq 0$, a contradiction.

The opposite implication is obvious. □

Proof of Theorem 2.7. An ultrafilter F on B corresponds uniquely to a $\{0,1\}$-valued measure μ_F defined by $\mu_F(x) = 1$ *iff* $x \in F$, and 0 otherwise.

Given the simple observation that $L = \{f \in B^\omega : (\forall \mu\ \{0,1\}\text{-}valued\ measure)\ \mu(f(n)) \to 0\}$ and 1.8 (ii), it follows that L is a zero-convergence structure, Urysohn closed, and moreover it is exhaustive.

In the following we will show any exhaustive submeasure μ on B is continuous in the topology determined by L.

Let $\langle x_n \rangle \in L$ and let $\varepsilon > 0$. We want to show that for some n_0, $\mu(x_n) \le \varepsilon$ whenever $n \ge n_0$. Using 2.8, by induction we can construct a $\varphi \in \omega^\omega\uparrow$ with the property that $\mu(\bigvee_{k \le i \le p} x_i - \bigvee_{k \le i \le \varphi(k)} x_i) < \varepsilon/2^k$.

Set $a_k = \bigvee_{k \le i \le \varphi(k)} x_i$. It follows that $x_k \le a_k$ and $x_k \le \bigwedge_{i \le k} a_i \vee \bigvee_{i \le k} (a_k - a_i)$.

Since for $i \le k$, $a_k - a_i = \bigvee_{k \le j \le \varphi(k)} x_j - \bigvee_{i \le j \le \varphi(i)} x_j$, $\mu(a_k - a_i) < \varepsilon/2^i$ and thus $\mu(\bigvee_{i \le k} (a_k - a_i)) < 2\varepsilon$.

Set $b_k = \bigwedge_{i \le k} a_i$. $\langle b_k \rangle$ is a descending sequence. If $\langle b_k \rangle$ is not it L, then $\langle b_k \rangle$ has a finite intersection property and hence can be extended to an ultrafilter F on B. Then for any k there is an $i \ge k$ so that $x_i \in F$, and so there is $\langle y_n \rangle$, a subsequence of $\langle x_n \rangle$, with $\mu_F(y_n) \to 1$, a contradiction with the definition of L. Thus $\langle b_k \rangle \to L$, and, consequently, for some k_0, $b_k = \mathbf{0}$ for any $k \ge k_0$, and therefore $\mu(b_k) \to 0$.

Since $x_k \le \bigwedge_{i \le k} a_i \vee \bigvee_{i \le k} (a_k - a_i)$, for sufficiently large k, $\mu(x_k) \le \mu(\bigvee_{i \le k} (a_k - a_i)) < 2\varepsilon$. □

The just presented proof motivates the following notion.

2.9 Definition. A zero-convergence structure $\mathscr{I}$ on B is *groupwise closed* if for every $\langle x_n : n \in \omega \rangle \in \mathscr{I}$ and every $\varphi \in \omega^\omega\uparrow$, the sequence $\langle a_n \rangle$, where $a_n = \bigwedge_{k \le n} \bigvee \{x_i : k \le i \le \varphi(k)\}$, belongs to $\mathscr{I}$.

The following is an example of a class of groupwise closed exhaustive zero-convergence structures.

Each family $\mathscr{F}$ of ultrafilters on B with the property that $\bigcup \mathscr{F} = B - \{\mathbf{0}\}$, or equivalently, $\mathscr{F}$ is a dense subset of the Stone space of B, induces an exhaustive groupwise closed zero-convergence structure

$$L(\mathscr{F})(B) = \{f \in B^\omega : (\forall F \in \mathscr{F})\ \mu_F(f(n)) \to 0\},$$

where μ_F is the usual 2-valued measure determined by the ultrafilter F. Observe that $L = L(Ult(B))$.

The following generalization is a consequence of the proof of Theorem 2.7.

2.10 Proposition. *For any groupwise closed zero-convergence structure $\mathcal{I}$ on B, an exhaustive submeasure μ is continuous in $\tau(\mathcal{I})$ if and only if $\mu(f(n)) \to 0$ for every decreasing $f \in \mathcal{I}$.*

2.11 Example. The order zero-convergence structure will be defined in two steps.

(i) for a σ-complete algebra C, set

$$Os(C) = \{f \in C^\omega : (\exists g \in C^\omega,\ g \downarrow \mathbf{0}_C)\ f \leq g\}.$$

(ii) for an arbitrary B, set $Os(B) = B^\omega \cap Os(C)$, where C is a σ-completion of B.

The convergence structure induced by the order zero-convergence structure on B is the most frequently studied one in the context of σ-fields of sets or σ-complete Boolean algebras.

2.12 Proposition. *For any algebra B, Os is an exhaustive groupwise closed zero-convergence, not necessarily Urysohn closed.*

Instead of a proof let us recall a few notions. Let C be a σ-completion of B.

For a sequence $\langle x_n : n \in \omega \rangle$ on C, we define limes superior, limes inferior, and limit as usual, i.e.

$$\overline{\lim}\, x_n = \bigwedge_{k \in \omega} \bigvee_{n \geq k} x_n \qquad \text{and} \qquad \underline{\lim}\, x_n = \bigvee_{k \in \omega} \bigwedge_{n \geq k} x_n,$$

$\lim x_n = x$ *if* $\overline{\lim}\, x_n = \underline{\lim}\, x_n = x$. Let us remark that $Os(C) = \{f \in C^\omega : \overline{\lim}\, f(n) = \mathbf{0}\}$. Since $\overline{\lim}(x_n \vee y_n) = \overline{\lim}\, x_n \vee \overline{\lim}\, y_n$ for arbitrary $\langle x_n \rangle, \langle y_n \rangle$, then $Os(C)$ is a zero-convergence. Since for any disjoint sequence $\langle x_n \rangle$, $\lim x_n = \mathbf{0}$, $Os(C)$ is exhaustive. $Os(B)$ has the same property; it follows from the fact that B is dense in C. The groupwise closedness follows from the definition of limes superior. Order zero-convergence structures for the Cantor algebra $\mathcal{A}$ and its completions are not Urysohn closed.

2.13 Example. Recall that zero-convergence structures are ordered by inclusion and that there are maximal ones. For an arbitrary algebra B, set

$$\mathcal{E}(B) = \bigcap \{\mathcal{I} : \mathcal{I} \textit{ is a maximal zero-convergence structure on } B\}.$$

$\mathcal{E}$ is again a zero-convergence structure, which is Urysohn closed, for it is an intersection of Urysohn closed structures. How large is it? In [Ja1] it is proven that $D \subseteq \mathcal{E}$, so in our terminology $\mathcal{E}$ is exhaustive. We will prove a little more; first we will describe a property determining which sequences belong to $\mathcal{E}$.

We are going to assign to each sequence $f \in B^\omega$ two elements from C, the completion of B.

For a complete Boolean algebra C, C^ω is again complete. For $f \in B^\omega$, let $Fil(f)$ denote the filter on B^ω generated by all subsequences of f and let $Idl(f)$ denote the ideal on B^ω generated by all subsequences. We allow in this context both, $Fil(f)$ and $Idl(f)$, to be improper. Put $v^+(f) = \bigwedge_{h \in Fil(f)} \bigvee_{n \in \omega} h(n)$ and $v^-(f) = \bigvee_{h \in Idl(f)} \bigwedge_{n \in \omega} h(n)$, where the joins and meets are taken in C.

2.14 Fact. (i) *both, v^+ and v^- are monotone on B^ω,*
(ii) *if g is a subsequence of f, then $v^+(g) \geq v^+(f)$ and $v^-(g) \leq v^-(f)$.*

2.15 Lemma. *A sequence $f \in B^\omega$ generates a zero-convergence structure on B if and only if $v^-(f) = \mathbf{0}$.*

Proof. It is evident that for any $f \in B^\omega$, the ideal $Idl(f) \subseteq \mathscr{Z}$ if and only if $v^-(f) = 0$. □

Set

$$Q(B) = \{f \in B^\omega : (\forall g \in Idl(f))\, v^+(g) = \mathbf{0}\}.$$

The following theorem provides the promised description and estimation of the size of $\mathscr{E}$.

2.16 Theorem. *For any algebra B, $Os \subseteq \mathscr{E} = Q$.*

2.17 Corollary. (Jakubík, [Ja1]) *Any zero-convergence structure can be extended to an exhaustive one.*

The proof of the theorem is devided into the two lemmas below.

2.18 Lemma. *$Q \subseteq \mathscr{E}$.*

Proof. (i) We show that for $f \in Q$, $Idl(f) \subseteq \mathscr{Z}$. If not, then there is $g \in Idl(f)$ with $\bigwedge\{g(n) : n \in \omega\} \geq a > \mathbf{0}$ for some $a \in B$. Then for every sequence $h \in Fil(g)$ we have $h \geq k_a$, therefore $v^+(g) > \mathbf{0}$, which is a contradiction with the definition of Q.
(ii) Assume $f \in Q$ and $\mathscr{I}$ is a maximal zero-convergence structure. Our aim is to show that $f \in Q$. From the maximality of I we have $f \in \mathscr{I}$ if and only if for any $h \in \mathscr{I}$ the ideal generated by $Idl(f) \vee \{h\}$ is a part of $\mathscr{Z}$.

Assume that $f \notin \mathscr{I}$ and so there is some $h \in \mathscr{I}$ such that the ideal generated by $Idl(f) \vee \{h\}$ is not a part of $\mathscr{Z}$. Then there is a $g \in Idl(f)$ such that $g \vee h \notin \mathscr{Z}$, i.e. there is an $a \in B^+$ and a subsequence q of $g \vee h$ so that $q \geq k_a$. Since both $Idl(f)$ and $\mathscr{I}$ are closed under subsequences and downward closed, we can pick g, h such that $g \vee h = k_a$ and $h = k_a - g$. Then necessarily $v^+(g) \geq a$. For any $g_0, \ldots, g_{m-1}$ subsequences of g, we have corresponding subsequences $h_0, \ldots, h_{m-1}$ of h and $\bigvee\{(g_0 \wedge \ldots \wedge g_{m-1})(n) : n \in \omega\} = a - \bigwedge\{(h_0 \vee \ldots \vee h_{m-1})(n) : n \in \omega\}$. Since $h \in \mathscr{I}$, so are $h_0, \ldots, h_{m-1}$, and therefore $\bigwedge\{(h_0 \vee \ldots \vee h_{m-1})(n) : n \in \omega\} = \mathbf{0}$. We showed that $v^+(g) = a \in B^+$, which is a contradiction. Thus $f \in \mathscr{I}$, hence $Q \subseteq \mathscr{I}$. □

2.19 Lemma. *If $f \in B^{\omega}$ and $v^{+}(f) > \mathbf{0}$, then $f \notin \mathscr{E}$.*

Proof. we assume that for a given $f, v^{+}(f) \geq a \in B^{+}$. Put $g = k_a - f$. We show that $v^{-}(g) = \mathbf{0}$, so g is contained in some maximal zero-convergence structure which cannot contain f since $f \vee g \geq k_a$. Let $g_0, \ldots, g_{m-1}$ be subsequences of g and $f_0, \ldots, f_{m-1}$ corresponding subsequences of f. Since $a - \bigwedge\{(g_0 \vee \ldots \vee g_{m-1})(n) : n \in \omega\} = \bigvee\{(f_0 \wedge \ldots \wedge f_{m-1})(n) : n \in \omega\} \geq a$, necessarily $\bigwedge\{(g_0 \vee \ldots \vee g_{m-1})(n) : n \in \omega\} = \mathbf{0}$. This proves that $v^{-}(g) = \mathbf{0}$. □

Proof of Theorem 2.16. (i) Verify that $Q = \mathscr{E}$. From Lemmas 2.18 and 2.19 we know that if $Idl(f)$ is a part of any maximal zero-convergence structure, then for any $g \in Idl(f)$, $v^{+}(g) = \mathbf{0}$, i.e. $f \in Q$. Since for $g \in Idl(f)$, $Idl(g) \subseteq Idl(f)$, if $f \in Q$, so is $Idl(f) \subseteq Q$. Thus $Q = \mathscr{E}$.
(ii) Verify that $Os \subseteq Q$. Let C be a completion of B and $f \in B^{\omega}$ so that $\overline{\lim} f = \mathbf{0}$. Then $Idl(f) \subseteq Os$ and for any $g \in Idl(f)$, $\bigwedge_{k \in \omega} \bigvee_{n \geq k} g(n) = \mathbf{0}$, and so $v^{+}(f) = \mathbf{0}$. This shows that $f \in Q$. □

Now we can summarize the various zero-convergent structures discussed up to now in the following diagram.

For any Boolean algebra B, and any family $\mathscr{F}$ of ultrafilters on B,

$$Fin \subseteq D \subseteq L \subseteq L(\mathscr{F}) \subseteq Os \subseteq \mathscr{E}.$$

All inclusions follow directly from the definitions of the corresponding zero-convergence structures, possibly except $L(\mathscr{F}) \subseteq Os \subseteq \mathscr{E}$. The latter of these two inclusions is proven in the previous theorem, and for the former it suffices to realize that $L(\mathscr{F})$ is groupwise closed and Os is the largest groupwise closed zero-convergence structure.

The following corollary is really a consequence of the notions and techniques introduced so far.

2.20 Corollary. *If $Conv_0(B)$ denotes the family of all zero-convergence structures on a Boolean algebra B, then $Conv_0(B)$ with the inclusion is a complete lattice if and only if $(\forall f \in B^{\omega})(v^{-}(f) = \mathbf{0} \Rightarrow v^{+}(f) = \mathbf{0})$.*

3. When $\mathscr{X}$ is a zero-convergence structure

In this section we answer our main question *when is the whole $\mathscr{X}$ a zero-convergence structure?*

3.1 Definition. We say that the Cantor algebra $\mathscr{A}$ is *almost regularly embedded* into a Boolean algebra B if there is $\mathscr{A}'$, a subalgebra of B, so that

(i) $\mathscr{A}'$ is isomorphic to $\mathscr{A}$, and
(ii) there is a set $\{x_n : n \in \omega\}$ of generators of $\mathscr{A}'$ such that for any infinite subset X of ω, $\bigvee_B\{x_n : n \in X\} = \mathbf{1}$ and $\bigwedge_B\{x_n : n \in X\} = \mathbf{0}$.

3.2 Theorem. *For any B the following are equivalent.*

(i) *$\mathscr{Z}$ is a zero-convergence structure,*

(ii) $\mathscr{Z} = Q$,

(iii) *for any $a \in B^+$, the Cantor algebra $\mathscr{A}$ cannot be almost regularly embedded into $B \restriction a$.*

Recall, that B is $(\omega, 2)$-distributive if for any sequence $\langle a_n : n \in \omega \rangle \in B^\omega$ and for any $b \in B^+$, there is a $c \leq b, c \neq \mathbf{0}$, such that for any $n \in \omega$, either $c \leq a_n$ or $c \wedge a_n = \mathbf{0}$. Thus an $(\omega, 2)$-distributive algebra satisfies (iii), and so Jakubík's result [Ja] that for an $(\omega, 2)$-distributive Boolean algebra, $\mathscr{Z}$ is a zero-convergence structure, follows as a direct consequence of the theorem.

Proof. (i) $\leftrightarrow$ (ii) is clear.

Proving *not (i) $\rightarrow$ not (iii).* $\mathscr{Z}$ is not a zero-convergence structure iff $\mathscr{Z}$ is not an ideal iff there are $a \in B^+$ and $f, g \in \mathscr{Z}$ such that $f \vee g = k_a$. Then $\{f(n) : n \in \omega\} \subseteq B \restriction a$. Let $\mathscr{A}'$ be a subalgebra of $B \restriction a$ generated by $\{f(n); n \in \omega\}$. Then $\mathscr{A}'$ is countable. Moreover, it is atomless, otherwise there is an atom $c \neq \mathbf{0}$ and so for any n, either $c \leq f(n)$ or $c \wedge f(n) = \mathbf{0}$. One of those cases must happen infinitely many times. The former contradicts the fact that $f \in \mathscr{Z}$, while the latter contradicts the fact that $g \in \mathscr{Z}$. Thus $\mathscr{A}'$ is isomorphic to the Cantor algebra.

For proving *not (iii) $\rightarrow$ not (i)*, set $f(n) = x_n$ and $g(n) = a - x_n$. It follows that $f \vee g = k_a$, hence $\mathscr{Z}$ is not an ideal. $\square$

In the following we will characterize some Boolean algebras that satisfy the theorem using their forcing properties. We shall explain how some of the notions discussed previously can be reinterpreted in terms of properties of reals in generic extensions and restated in the language of forcing.

Recall well-known basic relations concerning the interrelationship of functions and subsets of ω in a generic extension and the ground model. Let M denote a generic extension of V. $X \subseteq \omega$ in the extension is said to be an *independent* (or *splitting*) *real* over V if for all $Y \in [\omega]^\omega \cap V$ both $X \cap Y$ and $Y - X$ are infinite. A function $f \in M$, $f \in \omega^\omega$, is a *dominating real* over V iff for all $g \in \omega^\omega \cap V$ for all but finitely many $n \in \omega$, $g(n) \leq f(n)$. M is an *ω^ω-bounding extension* of V if every $f \in M$, $f \in \omega^\omega$ is bounded by a $g \in \omega^\omega \cap V$, i.e. $f(n) \leq g(n)$ for any n.

If B is a Boolean algebra and C its completion, sequences from C^ω can be viewed as canonical names for all reals in a generic extension when forcing with $(B^+, \leq)$ or $(C^+, \leq)$. Sequences from B^ω can be viewed as names for elements of a subfield of all reals in the generic extension. If G is a generic filter on C over V, then a real ($=$ subset of ω) in $V[G]$ named by $f \in C^\omega$ is $f_G = \{n : f(n) \in G\}$.

$\mathscr{Z}(B)$ is thus a set of names of reals and hence determines a set of reals in any generic extension. The question *which set of reals in a generic extension $\mathscr{Z}(B)$ determines?* has an answer in the following fact.

3.3 Fact. *For any $f \in B^\omega$, $f \in \mathscr{X}$ iff for any generic filter G on B, $f_G \in V[G]$ does not contain an infinite subset from V.*

In fact, more is easy to see.

3.4 Fact. (i) *For any generic filter G on B, $\mathscr{X}_G = \{f_g : f \in \mathscr{X}\}$ is a family of subsets of ω in $V[G]$ closed under taking subsets with the property that $\mathscr{X}_G \cap V = [\omega]^{<\omega}$.*

(ii) *$\mathscr{X}$ is a zero-convergence iff for any generic filter G on B over V, in $V[G]$, $\mathscr{X}_G$ is a proper ideal on ω.*

Similarly, the question *when $\mathscr{X}$ is a zero-convergence structure?* can be answered using the forcing properties of B.

3.5 Theorem. *$\mathscr{X}(B)$ is a zero-convergence structure iff for any generic filter G on B and any $f \in B^\omega$, f_G is not an independent real in $V[G]$.*

Proof. We shall prove the negated version of the equivalence, i.e. *$\mathscr{X}$ is not a zero-convergence structure iff there are $a \in B^+$ and $f \in B^\omega$ such that for any generic filter G on B containing a, f_G is an independent real in $V[G]$.*

Let assume that $\mathscr{X}$ is not an ideal. Take $f \in B^\omega$ and $a \in B^+$ so that $f \in \mathscr{X}$ and $g = k_a - f$ also belongs to $\mathscr{X}$. Let $X \subseteq \omega$ be infinite. Since $\bigvee_{n \in X} f(n) = a \in G$, it follows from the genericity of G that there is $n_0 \in X$ so that $f(n_0) \in G$. For any $k \in \omega$, the set $X_k = \{n \in X : n > k\}$ is again infinite and so $\bigvee_{n \in X_k} \in G$, therefore for some $n_1 > k$, $f(n_1) \in G$. This argument shows that $X \cap f_G$ is infinite. Since $g \in \mathscr{X}$ and $g_G = \omega - f_G$, it has the same property, $\bigvee_{n \in X} g(n) = a$ for any $x \in [\omega]^\omega$, and the same argument as for f shows that $X \cap g_G = X - f_g$ is infinite. Thus f_G is an independent real.

For the proof of the opposite implication, assume that for any generic filter G on B over V there is an $f \in B^\omega$ such that f_G is an independent real in $V[G]$. Consider an infinite subset X of ω from the ground model. Since $X \subseteq f_G$ does not hold, there is an $n \in X$ such that $f(n) \notin G$. Thus $\bigwedge_{n \in X} f(n) \notin G$. Let C be a completion of B. For the element $c = \overline{\lim} f(n) - \bigvee_{X \in V \cap [\omega]^\omega} \bigwedge_{n \in X} f(n)$ from C there is an $a \in B$ so that $a \leq c$ and $a \in G$. Set $\bar{f}(n) = f(n) \wedge a$. Then $\bar{f}_G \in B^\omega$, $\bar{f}_G = f_G$ and from the independence of f_G we get that $k_a - \bar{f} \in \mathscr{X}$ and $\bar{f} \in \mathscr{X}$. Thus $\mathscr{X}$ cannot be an ideal. □

Hence any forcing notion that does not add an independent real gives an example of a complete Boolean algebra B for which $\mathscr{X}(B)$ is a zero-convergence structure. Among them the ones that add a real, but not an independent real, are the non-trivial and interesting ones. There are several examples of such forcing notions. The most familiar are Sacks forcing ([Sa]), Miller forcing ([Mi]), Blass-Shelah forcing ([BS]), and Matet forcing ([B]). Therefore, Boolean algebras of regular open sets of these partial orders and all their dense subalgebras are examples of non $(\omega.2)$-distributive Boolean algebras for which $\mathscr{Z}$ is a zero-convergence structure.

It is well known that among the forcing notions mentioned above, only Sacks forcing is ω^ω-bounding. On the other hand, any forcing notion adding a dominating real adds also an independent real and so it is not an example of an algebra where $\mathscr{Z}$ is a zero-convergence structure.

In the following we shall provide a simple description of $\mathscr{Z}(B)$ for these examples.

3.6 Definition. Let B be a σ-complete Boolean algebra. A sequence $f \in B^\omega$ has an *absolute value* if for any subsequence g of f, $\overline{\lim}\, g(n) = \overline{\lim}\, f(n)$, $\underline{\lim}\, g(n) = \underline{\lim}\, f(n)$. In such a case, the value $a = \overline{\lim}\, f(n) - \underline{\lim}\, f(n)$ is called the *absolute value* of f and we denote it by abs lim $f(n) = a$.

Recall the definition of one of the standard cardinal invariants of the continuum, $\mathfrak{t}$, which is the smallest cardinality of a strictly decreasing chain of infinite subsets of ω (ordered by inclusion modulo finite sets) without an infinite lower bound.

The following Lemma is a generalization of a result from [Vl].

3.7 Lemma. *Let B be a σ-complete algebra satisfying the $\mathfrak{t}$-cc. Then for any $f \in B^\omega$, there is a subsequence g of f which has an absolute value.*

Proof. Note that subsequences of f are determined by subsets of ω. If g is a subsequence of f, then $\overline{\lim}\, g(n) \leq \overline{\lim}\, f(n)$ and $\underline{\lim}\, g(n) \geq \underline{\lim}\, f(n)$. For an $X \in [\omega]^\omega$, set $a_X = \overline{\lim}\{f(n) : n \in X\} - \underline{\lim}\{f(n) : n \in X\}$.

Assume that such subsequence g does not exist. Then we can construct a descending chain $\langle X_\alpha : \alpha < \mathfrak{t}\rangle$ of subsets of ω such that for $\alpha < \beta$, $|X_\alpha - X_\beta| = \omega$, and $|X_\beta - X_\alpha| < \omega$, and $a_{X_\alpha} > a_{X_\beta}$.

Then $\{a_{X_\alpha} - a_{X_{\alpha+1}} : \alpha < \mathfrak{t}\}$ is a disjoint family of size $\mathfrak{t}$, a contradiction. □

3.8 Theorem. *Let B be a Boolean algebra satisfying the $\mathfrak{t}$-cc and let $\mathscr{Z}$ be a zero-convergence structure. Then $\mathscr{Z} = \mathscr{U}(Os)$.*

Proof. We aim at proving that for any $f \in \mathscr{Z}$ there is a subsequence g of f with $\overline{\lim}\, g(n) = 0$. Fix $f \in \mathscr{Z}$. By Lemma 3.7, there is a g, a subsequence of f, with abs lim $g(n) = a$ for some $a \in B$. Since $g \in \mathscr{Z}$, abs lim $f(n) = \overline{\lim}\, g(n) = a$. Let us assume that $a \neq \mathbf{0}$. Then $h = k_a - g$ is in $\mathscr{Z}$ and so $\mathscr{Z}$ is not an ideal, which is a contradiction. □

The above examples of Boolean algebras of various forcing notions, for which $\mathscr{Z}$ is an ideal, do not satisfy the $\mathfrak{t}$-cc, though they all satisfy $(2^\omega)^+$-cc. An interesting open problem is whether there is a ccc complete atomless countably generated Boolean algebra B for which $\mathscr{Z}$ is a zero-convergence structure. The positive answer to the question is certainly equiconsistent with ZFC. For, consider $\mathscr{S}$, a complete Boolean algebra of regular open sets of Sacks forcing. It is known

[GJ] that $\mathscr{S}$ is a complete subalgebra of some algebra B, which is (σ-closed * ccc)-decomposable, i.e. there is a C, a complete subalgebra of B, such that C has a σ-closed dense subset and for any generic filter G on C, the following holds true in $V[G]$: when G is extended to $\tilde{G}$ on B by $\tilde{G} = \{a \in B : (\exists b \in C \cap G)(a \geq b)\}$, then the quotient algebra $B/\tilde{G}$ is atomless, countably generated, and satisfies the ccc. Since B does not add an independent real, in the extension, $\mathscr{Z}(B/\tilde{G})$ is a zero-convergence structure.

Remark. The sequential topolohy determined by Os is neither Hausdorff nor Fréchet for completion of any of the four forcing notions discussed above, see [BGJ].

We do not know of an example of an algebra for which $\mathscr{U}(Os) \neq \mathscr{E}$. We conclude with a conjecture. *For any Boolean algebra B, if $\mathscr{Z}(B)$ is a zero-convergence structure, then $\mathscr{Z}(B) = \mathscr{U}(Os(B))$.*

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