A Hedgehog in a Product

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We shall construct, under CH, two Fréchet-Urysohn α_4-spaces, the product of which is Fréchet-Urysohn, but fails to be α_4. This answers T. Nogura's question from 1985.

In 1972, A. V. Archangel'skij introduced the classes of α_i-spaces $(1 \leq i \leq 4)$, providing thereby a finer classification of Fréchet-Urysohn spaces. T. Nogura proved in 1985 [No] that for $i = 1, 2, 3$, the product of two α_i-spaces remains α_i, leaving the question for the α_4 spaces open. He gave an example of two α_4 Fréchet-Urysohn compact spaces such that their product is neither Fréchet-Urysohn nor α_4. These results, of course, led to two natural questions: If X and Y are Fréchet-Urysohn *and* α_4, and if $X \times Y$ is either α_4 or Fréchet-Urysohn, must it have the other property, too? For more information on the topics and an extensive bibliography, see also a survey paper by P. Nyikos [Ny].

Here we want to present an example of two Fréchet-Urysohn α_4 spaces, whose product remains Fréchet-Urysohn but fails to be α_4. It solves negatively Problem 3.15 from [No]. The construction is done under the Continuum Hypothesis and we have no idea concerning the ZFC example or even an example under some weakening of CH. The author feels indebted to Camillo Constantini for turning his attention to this topic.

The notation used throughout the paper is standard. If X is a set and κ is a cardinal, then $[X]^\kappa$ denotes the set $\{Y \subseteq X : |Y| = \kappa\}$, similarly for $[X]^{<\kappa}$. For two countable sets A, B, the almost inclusion $A \subseteq^* B$ means $|A \setminus B| < \omega$.

Definition. A space X is called *Fréchet-Urysohn* is for every set $C \subseteq X$ and every point $x \in \overline{C}$ there is a sequence $\langle x_n : n \in \omega \rangle$ ranging in C and converging to x.

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If X is a topological space and $\varphi : \omega \to X$ is a sequence ranging in X, let us simplify the notation and speak about the set $A = rng\ \varphi$ as about a sequence as well. This convention may lead to some difficulties if, say, the constant sequence is considered, but in general, advantages prevail in all cases when the sequence φ is finite-to-one. Thus, when we use the phrase "a sequence A converges to a point x" we mean that $A \subseteq X$, $|A| = \omega$, and, for every neighborhood G of x, the set $A \setminus G$ is finite.

Definition. A space X is called α_4, if for every $x \in X$ and every countable family $\{A_n : n \in \omega\}$ of sequences converging to x there is a sequence B converging to x such that $A_n \cap B \neq \emptyset$ for infinitely many $n \in \omega$.

Consider a countable hedgehog (a sequential fan, or S_ω in other terminology), i.e., the quotient space $(\omega + 1) \times \omega/\sim$, where $(x, y) \sim (x', y')$ if and only if $x = x' = \omega$. This is the simplest Fréchet-Urysohn space which is not α_4, moreover, it is a test space for the α_4 property: F. Siwiec proved that a Fréchet-Urysohn space is α_4 if and only if it does not contain a copy of S_ω [Si]. Our main theorem reads as follows.

Theorem. *Assume CH. Then there are Fréchet-Urysohn α_4-spaces X and Y such that $X \times Y$ is Fréchet-Urysohn and contains a copy of a hedgehog, hence $X \times Y$ is not α_4.*

We postpone the proof of the theorem, in order to prepare at first several tools for an easier presentation. It is a usual approach to give examples in this field as very simple topological spaces, namely, the spaces with just one non-isolated point. So suppose that X is a topological space, whose underlying set is $\omega \cup \{\infty_X\}$, where $\infty_X \notin \omega$ and ∞_X is the only non-isolated point of X. Denote by $\mathscr{F}(X)$ the filter $\{U \cap \omega : U \text{ is a neighborhood of } \infty_X\}$. With this notation, the next two lemmas are easy and perhaps known.

Lemma 1. *For a space X with a unique nonisolated point ∞_X, let $\mathscr{A}(X) \subseteq [\omega]^\omega$ be an arbitrary collection satisfying*

(a) *for every $A \in \mathscr{A}(X)$, A converges to ∞_X,*

(b) *any two distinct members of $\mathscr{A}(X)$ are almost disjoint,*

(c) *$\mathscr{A}(X)$ is a maximal family satisfying (a) and (b).*

Denote by $\mathscr{G}(X) = \{G \subseteq \omega : (\forall A \in \mathscr{A}(X))\, |A \setminus G| < \omega\}$. Then $\mathscr{F}(X) \subseteq \mathscr{G}(X)$. If the space X is Fréchet-Urysohn then $\mathscr{F}(X) = \mathscr{G}(X)$.

Proof. If $F \in \mathscr{F}(X)$ and $A \in \mathscr{A}(X)$, then by (a), $A \setminus F$ is finite, so $F \in \mathscr{G}(X)$. Suppose now that X is Fréchet-Urysohn and choose an arbitrary $G \in \mathscr{G}(X)$. It is enough to show that $\infty_X \notin \overline{\omega \setminus G}$. Suppose not, then, since X is Fréchet-Urysohn, there is some sequence $B \subseteq \omega \setminus G$ which converges to ∞_X. By maximality of $\mathscr{A}(X)$, there is some $A \in \mathscr{A}(X)$ with $|A \cap B| = \omega$, therefore $A \setminus G$ is infinite for this $A \in \mathscr{A}(X)$, which contradicts the definition of $\mathscr{G}(X)$. So $G \cup \{\infty_X\}$ is a neighborhood of ∞_X and $G \in \mathscr{F}(X)$. □

Knowing now that a Fréchet-Urysohn space with one nonisolated point can be fully described by a suitable almost disjoint family, let us translate the notion of α_4 to this setting.

Notation. Let $\mathscr{A} \subseteq [\omega]^\omega$ be an almost disjoint family. For $M \subseteq \omega$, denote

$$M \wedge\wedge \mathscr{A} = \{M \cap A : A \in \mathscr{A} \text{ and } |M \cap A| = \omega\}.$$

Next, let

$$\mathscr{I}^+(\mathscr{A}) = \{M \subseteq \omega : |M \wedge\wedge \mathscr{A}| \geq \omega\}.$$

Lemma 2. *Let X be a space with a unique nonisolated point, and let $\mathscr{A}(X)$ be as in Lemma 1. The space X is α_4 if and only if for each $M \in \mathscr{I}^+(\mathscr{A}(X))$, $M \wedge\wedge \mathscr{A}(X)$ is uncountable.*

Proof. Assume the condition holds. Let C_n converge to ∞_X for all $n \in \omega$. By maximality of $\mathscr{A}(X)$, for every n there is a set $A_n \in \mathscr{A}(X)$ so that $A_n \cap C_n$ is infinite. If there is some k such that $\{n \in \omega : |A_k \cap C_n| = \omega\}$ is infinite, then A_k is the sequence witnessing the α_4 property. If there is no such $k \in \omega$, put $M = \bigcup_{n\in\omega} C_n \cap A_n$. Then $M \in \mathscr{I}^+(\mathscr{A}(X))$ and so by the condition, there is some $A \in \mathscr{A}(X) \setminus \{A_n : n \in \omega\}$, with the intersection $A \cap M$ infinite. The sequence A converges to ∞_X and $A \cap C_n$ is non-empty for infinitely many C_n's, so α_4 is verified in this case, too.

Assume the condition fails. Pick $M \in \mathscr{I}^+(\mathscr{A}(X))$ so that $|M \wedge\wedge \mathscr{A}(X)| = \omega$. We are allowed to enumerate $M \wedge\wedge \mathscr{A}(X)$ as $\{M \cap A_n : n \in \omega\}$. Define inductively $B_0 = A_0 \cap M$, $B_n = A_n \cap M \setminus \bigcup_{k<n} A_k$. Then the family $\{B_n : n \in \omega\}$ consists of pairwise disjoint sequences converging to ∞_X. If there was a convergent sequence C such that $C \cap B_n$ is non-empty for infinitely many n's, then, thinning C a bit if necessary, we may assume that $C \cap A_n$ is always finite. By the maximality of $\mathscr{A}(X)$, there is some $A \in \mathscr{A}(X)$ with $|A \cap C| = \omega$. Since $C \subseteq M$, $A \cap M \in M \wedge\wedge \mathscr{A}(X)$, hence $A = A_n$ for some n. But this is absurd, because $A_n \cap B_m = \emptyset$ whenever $m > n$. So X is not α_4. □

Now, we are ready to prove the theorem. We have to find two spaces X and Y with the only nonsolated points ∞_X and ∞_Y, respectively. So we need to define the filters $\mathscr{F}$ $(= \mathscr{F}(X))$ and $\mathscr{G}$ $(= \mathscr{F}(Y))$ on ω, which is, according to Lemma 1, the same as to find two almost disjoint families $\mathscr{A}$ $(= \mathscr{A}(X))$ and $\mathscr{B}$ $(= \mathscr{A}(Y))$ on ω. It will turn out that we shall do somehow reduntantly both tasks jointly.

The construction will be done by a transfinite induction to ω_1. Before the start, let us introduce necessary bookkeeping. Using the Continuum Hypothesis, enumerate $[\omega]^\omega = \{M_\alpha : \alpha < \omega_1\}$, $\{C \in [\omega \times \omega]^\omega : C \cap \Delta = \emptyset\} = \{C_\alpha : \alpha < \omega_1\}$, ${}^\omega\omega = \{f_\alpha : \alpha < \omega\}$, and arrange the enumeration so that all items in the first two lists occur repeated ω_1-times. The diagonal Δ is, of course, the set $\Delta = \{(n,n) : n \in \omega\}$.

Fix some partition $\{S_n : n \in \omega\}$ of the set ω with each member S_n infinite; e.g., $S_n = \{2^n \cdot (2k+1) - 1 : k \in \omega\}$.

In each step of the inductiion, we shall define three sets R_α, F_α, G_α and two countable almost disjoint collections $\mathscr{A}_\alpha$, $\mathscr{B}_\alpha$ which will satisfy the following:

(i) For each $\alpha < \omega_1$ and for each $n \in \omega$, $S_n \subseteq^* R_\alpha$;

(ii) for each $\alpha < \omega_1$ and for each $A \in \mathscr{A}_\alpha$, $A \subseteq^* R_\alpha \cup F_\alpha$, while for each $B \in \mathscr{B}_\alpha$, $B \subseteq^* R_\alpha \cup G_\alpha$;

(iii) for each $\alpha < \omega_1$, $R_\alpha \cup F_\alpha \cup G_\alpha = \omega$, $R_\alpha \cap F_\alpha = R_\alpha \cap G_\alpha = F_\alpha \cap G_\alpha = \emptyset$;

(iv) for each $\alpha < \beta < \omega_1$, $R_\alpha \supseteq^* R_\beta$, $F_\alpha \subseteq^* F_\beta$, $G_\alpha \subseteq^* G_\beta$, $\mathscr{A}_\alpha \subseteq \mathscr{A}_\beta$, $\mathscr{B}_\alpha \subseteq \mathscr{B}_\beta$;

(v) for each $\alpha < \omega_1$, if $M_\alpha \in \mathscr{I}^+(\mathscr{A}_\alpha)$, then there is a set $A \in \mathscr{A}_{\alpha+1} \setminus \mathscr{A}_\alpha$ with $A \subseteq M_\alpha$; analogously for $\mathscr{B}_\alpha$;

(vi) for each $\alpha \in \omega$, if for every finite set $L \subset \omega$ the set $C_\alpha \cap (R_\alpha \cup F_\alpha \setminus L) \times (R_\alpha \cup G_\alpha \setminus L)$ is infinite, then there is a set $A = \{a_n : n \in \omega\} \in \mathscr{A}_{\alpha+1}$ and a set $B = \{b_n : n \in \omega\} \in \mathscr{B}_{\alpha+1}$ such that for some infinite set $Q \subseteq \omega$ one has $\{(a_n, b_n) : n \in Q\} \subseteq C_\alpha$;

(vii) for each $\alpha \in \omega$, $F_{\alpha+1} \cup G_{\alpha+1} \supseteq \{k \in \omega : \text{for some } n \in \omega,\ k \in S_n \ \&\ k \leq f_\alpha(n)\}$.

Case $\alpha = 0$:

Put simply $R_0 = \omega$, $F_0 = G_0 = \emptyset$, $\mathscr{A}_0 = \mathscr{B}_0 = \{S_n : n \in \omega\}$.

Let $\alpha < \omega_1$ and suppose that R_β, F_β, G_β, $\mathscr{A}_\beta$, $\mathscr{B}_\beta$ have been already found for all $\beta < \alpha$.

Case $\alpha = \beta + 1 < \omega_1$.

The easiest point is to guarantee the actual instance of (vii), so let us start with it. Denote by H the set $\{k \in \omega : \text{for some } n \in \omega,\ k \in S_n \ \&\ k \leq f_\beta(n)\} \setminus (F_\beta \cup G_\beta)$. Let $F' = F_\beta \cup H$, $G' = G_\beta$.

Now we shall take care of (v). Suppose $M_\beta \in \mathscr{I}^+(\mathscr{A}_\beta)$. Then, because of (ii), the set $M_\beta \cap (F' \cup R_\beta)$ belongs to $\mathscr{I}^+(\mathscr{A}_\beta)$ as well. However, the almost disjoint collection $\mathscr{A}_\beta$ is countable only, so there is an infinite set $A_1 \subseteq M_\beta \cap (F' \cup R_\beta)$ which is almost disjoint with all $A \in \mathscr{A}_\beta$. Observe that this implies that $A_1 \cap S_n$ is finite for all $n \in \omega$, because every S_n is a member of $\mathscr{A}_0 \subseteq \mathscr{A}_\beta$. Put $F'' = F' \cup A_1$ and $\mathscr{A}' = \mathscr{A}_\beta \cup \{A_1\}$.

If $M_\beta \notin \mathscr{I}^+(\mathscr{A}_\beta)$, then let $F'' = F'$, $\mathscr{A}' = \mathscr{A}_\beta$.

The same reasoning allows us to find a set B_1 contained in $(\omega \setminus F'') \cap M_\beta$ and almost disjoint with every $B \in \mathscr{B}_\beta$. Put $G'' = G' \cup B_1$ and $\mathscr{B}' = \mathscr{B}_\beta \cup \{B_1\}$. Similarly as before, we shall relax if $M_\beta \notin \mathscr{I}^+(\mathscr{B}_\beta)$: We put $G'' = G'$ and $\mathscr{B}' = \mathscr{B}_\beta$ then.

Finally, suppose that for every finite set L, the intersection $C_\beta \cap (R_\beta \cup F_\beta \setminus L) \times (R_\beta \cup G_\beta \setminus L)$ is infinite. Proceeding by an induction, we can easily find integers a_n, b_n such that $(a_n, b_n) \in C_\beta \cap (R_\beta \cup F_\beta \setminus L_n) \times (R_\beta \cup G_\beta \setminus L_n)$, where $L_n = \{k, a_k, b_k : k < n\}$. Obviously, for the sets $A' = \{a_n : n \in \omega\}$ and $B' = \{b_n : n \in \omega\}$ we have $A' \subseteq R_\beta \cup F_\beta$ and $B' \subseteq R_\beta \cup G_\beta$ and $\{(a_n, b_n) : n \in \omega\} \subseteq C_\beta$. Notice moreover that the sets A' and B' are disjoint because of our choice of the sets L_n and by the fact that $C_\beta \cap \Delta = \emptyset$. If the set A' is almost disjoint with all members from $\mathscr{A}'$, then put $A_2 = A'$ and $Q' = \omega$. Otherwise select an arbitrary set $A \in \mathscr{A}'$ for which $|A \cap A'| = \omega$, put $Q' = \{n \in \omega : a_n \in A\}$ and leave the set A_2 to be undefined. Next, apply the same reasoning onto $B'' = \{b_n : n \in Q'\}$: Either B'' is almost disjoint with all members of

$\mathscr{B}'$, then let $Q = Q'$ and $B_2 = B''$, or there is some $B \in \mathscr{B}'$ with $|B \cap B''| = \omega$, in which case we put $Q = \{n \in Q' : b_n \in B\}$ and the set B_2 is undefined then.

If the set A_2 has been already defined, then put $F_\alpha = F'' \cup A_2$ and $\mathscr{A}_\alpha = \mathscr{A}' \cup \{A_2\}$; if has not been defined, then $F_\alpha = F''$ and $\mathscr{A}_\alpha = \mathscr{A}'$. The set G_α and the family $\mathscr{B}_\alpha$ are defined analogously.

It remains to complete the inductive definition by putting $R_\alpha = \omega \setminus (F_\alpha \cup G_\alpha)$.

Case $\alpha < \omega_1$, α limit:

Here we need to take care on (i), (ii), (iii) and (iv) only. Choose a mapping $h : \omega \to \omega$ in such a way that for all $\beta < \alpha$ we have $F_\beta \cup G_\beta \subseteq^* \{k \in \omega : \text{for some } n \in \omega,\ k \in S_n \ \&\ k \leq h(n)\}$. This is clearly possible, because α is countable and for all $n \in \omega$ and all $\beta < \alpha$, $S_n \cap (F_\beta \cup G_\beta)$ is finite by (i) and (iii). Put $R_\alpha = \omega \setminus \{k \in \omega : \text{for some } n \in \omega,\ k \in S_n \ \&\ k \leq h(n)\}$. Next, separate the countable family $\{F_\beta \setminus R_\alpha : \beta < \alpha\}$ from the family $\{G_\beta \setminus R_\alpha : \beta < \alpha\}$ by some set W, i.e., $W \supseteq^* F_\beta \setminus R_\alpha$ and $\omega \setminus W \supseteq^* G_\beta \setminus R_\alpha$ for all $\beta < \alpha$. It remains to put $F_\alpha = W \setminus R_\alpha$, $G_\alpha = \omega \setminus (R_\alpha \cup F_\alpha)$.

The description of all steps in the transfinite induction is complete.

It should be clear directly from the inductive definitions that the resulting R_α, F_α, G_α, $\mathscr{A}_\alpha$ and $\mathscr{B}_\alpha$ $(\alpha < \omega_1)$ satisfy (i)–(vii). Put $\mathscr{A} = \bigcup_{\alpha<\omega_1} \mathscr{A}_\alpha$, $\mathscr{B} = \bigcup_{\alpha<\omega_1} \mathscr{B}_\alpha$.

Let $X = \omega \cup \{\infty_X\}$ ($Y = \omega \cup \{\infty_Y\}$, resp.) be the space described in accordance with Lemma 1 by the almost disjoint family $\mathscr{A}$ ($\mathscr{B}$, resp.), i.e., a set $C \subseteq \omega$ converges to ∞_X if and only if $C \cap A$ is infinite for some $A \in \mathscr{A}$, and analogously for Y. Both spaces are obviously Fréchet-Urysohn. By (v) and by Lemma 2, they are also α_4: If $M \in \mathscr{I}^+(\mathscr{A})$, then there is some $\alpha < \omega_1$ such that $M \in \mathscr{I}^+(\mathscr{A}_\alpha)$, too. The set of indices $I = \{\beta < \omega_1 : M = M_\beta\}$ is uncountable, and for every $\beta \in I$, $\beta > \alpha$, there is some member of $\mathscr{A}_{\beta+1} \setminus \mathscr{A}_\beta$ contained in M. So $|M \wedge\wedge \mathscr{A}| \geq |I \setminus \alpha| = \omega_1$. The same reasoning applies for $\mathscr{B}$, too.

The following information will help us to show that $X \times Y$ is Fréchet-Urysohn:

Claim. *The family $\{\{\infty_X\} \cup R_\alpha \cup F_\alpha \setminus L : \alpha < \omega_1\ L \in [\omega]^{<\omega}\}$ is a neighborhood basis at ∞_X in X and the family $\{\{\infty_Y\} \cup R_\alpha \cup G_\alpha \setminus L : \alpha < \omega_1,\ L \in [\omega]^{<\omega}\}$ is a neighborhood basis at ∞_Y in Y.*

Proof of the claim. We shall prove the statement for X only, leaving the symmetrical argument to the reader. Fix an $\alpha < \omega_1$ and let $A \in \mathscr{A}$. If $A \in \mathscr{A}_\alpha$, then $A \subseteq^* R_\alpha \cup F_\alpha$ by (ii). If $A \notin \mathscr{A}_\alpha$, then $A \in \mathscr{A}_\beta$ for some $\beta > \alpha$. Thus $A \subseteq^* R_\beta \cup F_\beta$ by (ii), hence $|A \cap G_\beta| < \omega$ by (iii). Since $G_\alpha \subseteq^* G_\beta$ by (iv), $|A \cap G_\alpha| < \omega$ as well. This immediately implies that $A \subseteq^* R_\alpha \cup F_\alpha$ by (iii). Thus every set $\{\infty_X\} \cup R_\alpha \cup F_\alpha \setminus L$ for a finite $L \subseteq \omega$ is a neghborhood of ∞_X by Lemma 1.

Let U be an arbitrary neighborhood of a point ∞_X. We need to find some $\alpha < \omega_1$ such that $R_\alpha \cup F_\alpha \subseteq^* U \cap \omega$. Define a mapping $f \in {}^\omega\omega$ by the rule $f(n) = \min\{k \in \omega : (\forall j \in S_n \setminus U)\, j < k\}$. The mapping f was listed as $f = f_\alpha$ and by (vii) and (iii), $R_{\alpha+1} \subseteq^* U$. It clearly suffices to show that $F_{\alpha+1} \subseteq^* U$, too.

Suppose the contrary, let $H = F_{\alpha+1} \setminus U$ be infinite. Then by (iv), $H \cap F_\beta$ is infinite whenever $\beta > \alpha$. There is some ordinal $\beta > \alpha$ with $C_\beta = H \times \omega \setminus \Delta$. Since the set C_β obviously satisfies the assumptions of (vi), there is some $A \in \mathscr{A}_{\beta+1}$ satisfying its conclusion, which means in particular that $|A \cap H| = \omega$. But this contradicts the assumption that U is a neighborhood of ∞_X, because the sequence A converges to ∞_X and still $A \setminus U$ contains infinite set $H \cap A$. So $R_{\alpha+1} \cup F_{\alpha+1} \subseteq^* U$ and the claim is proved.

The product space $X \times Y$ is Fréchet-Urysohn. Let $C \subseteq X \times Y$, $x \in \bar{C}$. Since the subspaces $\{p\} \times Y$ for $p \in X$ ($X \times \{p\}$ for $p \in Y$, resp.) are homeomorphic to the Fréchet-Urysohn space Y (X, resp.), the only interesting case occurs when $C \subseteq \omega \times \omega$, $x = (\infty_X, \infty_Y)$. We shall suppose so for the rest.

If $(\infty_X, \infty_Y) \in \overline{C \setminus \Delta}$, then there is some $\alpha < \omega_1$ with $C_\alpha = C \setminus \Delta$. Then (vi) clearly gives a sequence ranging in C and converging to (∞_X, ∞_Y).

If $(\infty_X, \infty_Y) \in \overline{C \cap \Delta}$, the existence of a convergent sequence will be clear from the remaining part of the proof.

The subspace $\Delta \cup \{(\infty_X, \infty_Y)\} \subseteq X \times Y$ is homeomorphic to a hedgehog.

Indeed, since for every $n \in \omega$, $S_n \in \mathscr{A} \cap \mathscr{B}$, every set $\{(k,k) : k \in S_n\} \cup \{(\infty_X, \infty_Y)\}$ is homeomorphic to $\omega + 1$. The disjointness of the S_n's implies the disjointness of the $\{(k,k) : k \in S_n\}$'s. Whenever $f \in {}^\omega\omega$, then the set $\{(k,k) \in \Delta : \text{if } k \in S_n, \text{ then } k \le f(n)\}$ is closed discrete in $\Delta \cup \{(\infty_X, \infty_Y)\}$. To see this, choose an $\alpha < \omega_1$ with $f = f_\alpha$. Consider the set $U = (R_{\alpha+1} \cup F_{\alpha+1}) \times (R_{\alpha+1} \cup G_{\alpha+1})$. Then $U \cup \{(\infty_X, \infty_Y)\}$ is a neighborhood of (∞_X, ∞_Y). By (vii) and by (iii), whenever $k \in S_n$ and $k \le f_\alpha(n)$, then $k \notin R_{\alpha+1}$, so $k \in F_{\alpha+1} \cup G_{\alpha+1}$. But $F_{\alpha+1} \cap G_{\alpha+1} = \emptyset$, therefore $(k, k) \notin U$. □

Concluding remarks. The author's idea for the construction of this example was to build the spaces so that the required copy of S_ω would be the set $\Delta \cup \{(\infty_X, \infty_Y)\}$. Then the necessity to get a gap $\{F_\alpha, G_\alpha\}_\alpha$ became clear very soon. Since ω_1 is the only natural length of a gap, and since the character of a countable hedgehog is $\mathfrak{d}$, the assumtion $\mathfrak{d} = \omega_1$ seemed to be obligatory, too. But to take care of the Fréchet-Urysohn property needs to consider $\mathfrak{c}$ many subsets in the product, which together with the previous opted for CH as an assumption. Thus we strongly believe in an affirmative answer for this weakening of Nogura's problem:

Is it consistent that for two α_4 Fréchet-Urysohn spaces X, Y, the product $X \times Y$ is α_4 provided it is Fréchet-Urysohn?

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