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A Covering Property of Some Classes of Sets in $\mathbb{R}^2$

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We prove that if $\mathscr{B}$ is a class of open bounded subsets of $\mathbf{R}^n$ satisfying a simple geometric condition then the following Besicovitch-type covering property is true. For any ε there exists an M such that from any subclass $\mathscr{R} \subset \mathscr{B}$ one can select M subclasses of disjoint sets such that the selected sets cover at least the $1 - \varepsilon$ part of $\bigcup \mathscr{R}$.

Thus we get sufficient geometric condition for the minimal density property and for the CV_q covering properties introduced in [2].

During the proof we also get a reverse isoperimetric inequality for the union of star-shaped sets.

1. The result

In this note we prove a covering result (Theorem 3) that can be interesting in itself but also has connection with the following recently defined notions [2]. (Throughout the paper $|A|$ denotes the (Lebesgue) measure of A.)

Definition 1. Let $\mathscr{B}$ be a class of nonempty open bounded subsets of $\mathbf{R}^n$.

The class $\mathscr{B}$ is said to have the *minimal density property* (MDP) if there exists a function $\varrho : \mathbf{R}^+ \to \mathbf{R}^+$ such that if $H \subset \mathbf{R}^n$ is measurable with finite measure, $\mathscr{R} \subset \mathscr{B}$ covers H and the density of H in $\bigcup \mathscr{R}$ is $d > 0$ then one can find an $R \in \mathscr{R}$ in which the density of H is greater than $\varrho(d)$; that is,

$$\frac{|R \cap H|}{|R|} > \varrho\left(\frac{|H|}{|\bigcup \mathscr{R}|}\right).$$

The class $\mathscr{B}$ is said to have the *complete covering property* V_q (CV_q) for a fixed $1 \leq q \leq \infty$ if there exists a function $C : \mathbf{R}^+ \to \mathbf{R}^+$ such that for any $\varepsilon > 0$ and $\mathscr{R} \subset \mathscr{B}$ with $|\bigcup \mathscr{R}| < \infty$ we can find $R_1, \ldots, R_m \in \mathscr{R}$ such that

$$\text{(i)} \quad |\textstyle\bigcup_{k=1}^m R_k| \geq (1 - \varepsilon)\, |\cup \mathscr{R}| \quad \text{and} \quad \text{(ii)} \quad \left\| \sum_{k=1}^m \chi_{R_k} \right\|_q \leq C(\varepsilon)\, |\cup \mathscr{R}|^{1/q},$$

where $\|.\|_q$ denotes the L_q norm and χ_R is the characteristic function of R.

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Note that CV_q implies $CV_{q'}$ if $q > q'$. It is proved in [2] that MDP and CV_1 are equivalent and also that MDP implies that CV_q is equivalent with the classical (and weaker) covering property V_q for any $1 \le q < \infty$. (The covering property V_q is defined in [1], where – among others – the authors proved that V_q is equivalent with the weak type (p, p) property of the maximal operator associated to $\mathscr{B}$ if $1 < q < \infty$ and $1/p + 1/p = 1$.)

Unfortunately, it is not easy to prove the minimal density property (even for the simplest classes, like the class of balls), which makes the applicability of this notion harder. (In [2] the MDP is proved only for the class of intervals of $\mathbf{R}^n$ (i.e. n-dimensional axis-parallel rectangles)). It would be useful and interesting to have a weak sufficient geometric condition that guarantees the MDP. One can check (see [2] Example 4.7) that the class of sets in the plane that are the union of an open disc and an open sector with the same center and twice larger radius does not have the minimal density property. However, this is a regular class of sets (see Definition 7), which shows that the standard properties (regularity, V_q property (even for $q = \infty$), weak (1,1) property of the maximal operator, density property, differentiating properties) cannot guarantee the MDP. In this example the too "sharp" "thorn" is the obstacle of the MDP.

Our main result is the theorem below that shows that if the sets of $\mathscr{B}$ are "non-thorny" in the below defined sense then $\mathscr{B}$ has a much stronger property than the MDP or the CV_q properties: instead of (ii) of Definition 1, in this case, we have a better (Besicovitch type) control for the overlapping.

Definition 2. By a *drop* we mean the interior of the convex hull of a ball and a point (not contained in the ball). The angle of the drop is the angle between the line through the point and the center of the ball and any tangent line.

Let $0 < d < 1$ and $0 < \alpha < \pi/2$. We say that a bounded open set $H \subset \mathbf{R}^n$ is (d, α)-*non-thorny* if H is the union of drops with angle at least α and diameter at least $d \cdot \operatorname{diam} H$.

Theorem 3. *Let $\mathscr{R}$ be a family of (d, α)-non-thorny sets in $\mathbf{R}^n$ with bounded diameter. Then for any $\varepsilon > 0$ one can choose sets $R_1, \ldots, R_m \in \mathscr{R}$ such that*

(i)
$$\left|\bigcup_{k=1}^{m} R_k\right| \ge (1 - \varepsilon) \left|\cup \mathscr{R}\right| \quad \textit{and}$$

(ii) the sequence $R_1, \ldots, R_m$ can be distributed in M families of disjoint set, where M depends only on n, d, α and ε.

Remark 4. This covering property is similar to the Besicovitch property, the only difference is that, instead of all the centers, we cover a big part of the union. But, as the earlier mentioned example showed, in our case the Besicovitch property itself is not enough. However, we shall use the classical Besicovitch covering theorem (for balls) in the proof but we will also need estimate for the "edge" of the union of drops. This estimate will give us a reverse isoperimetric inequality for the union of star-shaped sets (Corollary 12), which can be interesting in itself.

Corollary 5. *For any* $0 < d < 1$ *and* $0 < \alpha < \pi/2$, *any class of* (d, A)*-non-thorny sets in* $\mathbf{R}^n$ *has the* CV_∞ *property and consequently the* CV_q *property for any* $1 \leq q \leq \infty$ *and the minimal density property as well.*

Therefore this non-thornity is a sufficient condition for the MDP but it is in fact too strong. However, as we shall see below, quite large and important classes satisfy it.

Definition 6. A set $H \subset \mathbf{R}^n$ is said to be *star-shaped at* x if $\overline{xy} \subset H$ for every $y \in H$, where $\overline{xy}$ denotes the closed segment between x and y.

The *hub of* H (hub(H)) is the set of all points at which H is star-shaped.

Let $r > 0$. We say that H is r*-star-shaped* if hub(H) contains an open ball with radius $r \cdot \operatorname{diam} H$.

Definition 7. A set $H \subset \mathbf{R}^n$ is r*-regular* if there exists a cube Q that contains H such that $|H|/|Q| > r$.

It is not hard to see (and probably well-known) that if H is a convex r-regular set in R^n then H is r'-star-shaped, where r' depends only on n and r. It is easy to see that any r-star-shaped set is (d, α)-non-thorny, where d and α depend only on r. Thus Theorem 3 has the following consequences:

Corollary 8. *If* $\mathcal{R}$ *is a class of convex r-regular open sets or a class of r-star-shaped open sets then for any* $\varepsilon > 0$ *one can select M subclasses of disjoint sets such that the selected sets cover the* $1 - \varepsilon$ *part of* $\bigcup \mathcal{R}$, *where M depends only on n, r and* ε.

Corollary 9. *Any class of convex r-regular open sets or of r-star-shaped open sets in* $\mathbf{R}^n$ *has the* CV_∞ *property and consequently the* CV_q *property for any* $1 \leq q < \infty$ *and the minimal density property as well.*

2. The proof of the result

Notation 10. Let $x \in \mathbf{R}^n$, $H \subset \mathbf{R}^n$ and $\delta > 0$. Let $S(x, \delta)$ denote the open ball with center x and radius δ. We denote the open neighborhood of H with radius δ by $S(H, \delta)$; that is,

$$S(H, \delta) = \bigcup_{x \in H} S(x, \delta) .$$

We also introduce the δ-interior by the following definition:

$$\operatorname{int}(H, \delta) = \{x : S(x, \delta) \subset H\} .$$

We denote the diameter of a set H by $\operatorname{diam}(H)$.

Lemma 11. *Let $\mathscr{H}$ be a family of r-star-shaped sets in $\mathbf{R}^n$ with diameter D and let $A = \bigcup \mathscr{H}$. Then for any $\delta \le D$ we have*

$$|S(A, \delta) \setminus A| \le C(n, r) \frac{\delta}{D} |A|, \tag{1}$$

where $C(n, r)$ depends only on n and r. (In fact, we can choose $C(n, r) = \left(\left(1 + \frac{\sqrt{n}+1}{r}\right)^n - 1\right)\left(\frac{\sqrt{n}+1}{r}\right)^n$.)

Proof. By homogeneity we can assume that $D = 1$.

For any $H \in \mathscr{H}$ there exists a ball $S(O_H, r) \subset \mathrm{hub}(H)$. Consider a cubic lattice with side $\frac{2r}{\sqrt{n}+1}$ and for a lattice point P let $S_P = S\left(P, \frac{r}{\sqrt{n}+1}\right)$. Let P_H be the nearest lattice point to O_H. Clearly, $P_H O_H \le \sqrt{n}\frac{r}{\sqrt{n}+1}$, so $S_{P_H} \subset S(O_H, r) \subset \mathrm{hub}(H)$. On the other hand, the balls S_P are disjoint.

For a lattice point P let

$$K_P = \bigcup \{H \in \mathscr{H} : S_P \subset \mathrm{hub}(H)\}.$$

Then $A = \bigcup \mathscr{H} = \bigcup_P K_P$ and only for those P for which $K_P \neq \emptyset$ we have $K_P \subset S(P, 1)$.

One can show (see e.g. [3] p. 286) that if $K \subset S(P, 1)$ and $S(P, a) \subset \mathrm{hub}(K)$ then the magnification of K with center P and ratio $1 + (\delta/a)$ contains $S(K, \delta)$. Then clearly

$$|S(K, \delta) \setminus K| \le \left(\left(1 + \frac{\delta}{a}\right)^n - 1\right)|K| \le \left(\left(1 + \frac{\delta}{a}\right)^n - 1\right)|s(0, 1)|.$$

Therefore in our case we have

$$|S(K_P, \delta) \setminus K_P| \le \left(\left(1 + \frac{\delta}{r/(\sqrt{n}+1)}\right)^n - 1\right)|S(0, 1)|.$$

Thus, denoting by N the number of those lattice points P for which K_P is nonempty, we have

$$|S(A, \delta) \setminus A| \le \sum_P |S(K_P, \delta) \setminus K_P| \le N\left(\left(1 + \frac{\delta(\sqrt{n}+1)}{r}\right)^n - 1\right)|S(0, 1)|.$$

On the other hand the balls S_P are disjoint subsets of A, hence

$$|A| \ge \sum_P |S_P| = N\left(\frac{r}{\sqrt{n}+1}\right)^n |S(0, 1)|.$$

Therefore, using that $\delta \le D = 1$, we get

$$\frac{|S(A,\delta)\setminus A|}{\delta|A|} \le \left(\frac{\sqrt{n}+1}{r}\right)^n \frac{\left(\left(1+\frac{\delta(\sqrt{n}+1)}{r}\right)^n - 1\right)}{\delta}$$
$$\le \left(\frac{\sqrt{n}+1}{r}\right)^n \left(\left(1+\frac{\sqrt{n}+1}{r}\right)^n - 1\right) = C(n,r). \qquad \square$$

Corollary 12. *If E is the union of r-star-shaped sets in* $\mathbf{R}^n$ *with diameter D then we have*

$$\frac{\bar{A}_+(E)}{|E|} \le \frac{C(n,r)}{D},$$

where $\bar{A}_+(E)$ *denotes the upper outer surface area in the sense of Minkowski, that is*

$$\bar{A}_+(E) = \limsup_{\delta\to 0+} \frac{|S(E,\delta)| - |E|}{\delta}. \qquad \square$$

Remark 13. If the diameters are not the same but between D_1 and D_2 then the same proof gives $\bar{A}_+(E)/|E| \le C(n, rD_1/D_2)/D_2$.

Remark 14. As a special case of Corollary 12, for example, we have that the ratio of the perimeter and the area of any finite union of (not necessary axis-parallel) unit squares is at most an absolute constant.

The author does not know the best constant. Is it 4?

Facts 15. *Let D be a drop (see Definition 2) with angle* $0 < \alpha < \pi/2$ *and with* $\operatorname{diam} D = d$. *Let* $E_\alpha = \frac{1}{\sin\alpha} + 1$ *and* $\delta < d/E_\alpha$. *Then*

1. *the radius of the "ball part" of D is* D/E_α.
2. *the set D is* $1/E_\alpha$*-star-shaped,*
3. *the set* $\operatorname{int}(D,\delta)$ *(see Notation 10) is a drop with angle* α *and with diameter* $d - E_\alpha\delta$,
4. *we have* $S(\operatorname{int}(D,\delta), E_\alpha\delta) \supset S(D,\delta)$ *and*
5. *for any* $0 < d' < d$ *and* $0 < \alpha' < \alpha$, *D can be written as the union of drops with angle* α' *and diameter* d'. $\square$

Lemma 16. *Let* $\mathscr{K}$ *be a family of* (d,α)*-non-thorny (see Definition 2) sets in* $\mathbf{R}^n$ *with diameter between* Δ *and* 2Δ, *let* $K = \bigcup\mathscr{K}$ *and let* $\delta \le d/2E_\alpha$. *Then one can choose sets* $K_1, \ldots, K_m \in \mathscr{K}$ *such that*

$$|S(K,\delta\Delta)\setminus \textstyle\bigcup_{k=1}^m K_k| \le C\delta|K|, \tag{2}$$

and the sequence $K_1, \ldots, K_m$ *can be distributed in* $M(\delta)$ *families of disjoint sets, where C depends only on n, d and* α *and* $M(\delta)$ *depends only on n and* δ.

Proof. By homogeneity, we can assume that $\Delta = 1$.

Let $\mathscr{D}$ be the family of those drops with diameter d and angle α that are contained in at least one of the sets of $\mathscr{K}$. Let $\mathscr{B}$ consist of the balls with radius δ contained in any drop of $\mathscr{D}$. Put

$$\mathscr{D}^* = \{\mathrm{int}(D,\delta) : D \in \mathscr{D}\} \quad \text{and} \quad K^* = \bigcup \mathscr{D}^* .$$

Note that, by definition and Fact 15.5, $K = \bigcup \mathscr{K} = \bigcup \mathscr{D}$ and that K^* is covered by the centers of the balls of $\mathscr{B}$. Thus, applying the classical covering theorem of Besicovitch, we get balls $B_1, \ldots, B_m \in \mathscr{B}$ that cover K^* but no point of $\mathbf{R}^n$ is covered more than C_n times. For $k = 1, \ldots, m$ let K_k be one of the sets of $\mathscr{K}$ that contain B_k. Then we have $\bigcup K_m \supset K^*$.

We claim that every set K_k intersects at most $C_n(4/\delta)^n$ sets of the sequence $K_1, \ldots, K_m$ (including itself). Indeed, for a fixed k the sets K_i, that intersect K_k, are contained in a ball with radius 4 (since each set has diameter at most 2), but on the other hand, they contain balls with radius δ that cover each point at most C_n times, hence the number of sets that intersect K_k is at most $C_n|S(0,4)|/|S(0,\delta)| = C_n(4/\delta)^n$.

Thus the sequence $K_1, \ldots, K_m$ can clearly be distributed in $M(\delta) = C_n(4/\delta)^n$ families of disjoint sets: the greedy algorithm easily gives a proper distribution.

Now we prove (2). Using Fact 15.4 we get

$$S(K^*, E_\alpha\delta) = \bigcup_{D\in\mathscr{D}} S(\mathrm{int}(D,\delta), E_\alpha\delta) \supset \bigcup_{D\in\mathscr{D}} S(D,\delta) = S\left(\bigcup\mathscr{D}, \delta\right) = S(K,\delta) .$$

Thus, using that $\bigcup_{k=1}^m K_k \supset K^*$, we have

$$S(K,\delta) \setminus \bigcup_{k=1}^m K_k \subset S(K^*, E_\alpha\delta) \setminus K^* .$$

According to Facts 15.2 and 15.3, $\mathscr{D}^*$ consists of $1/E_\alpha$-star-shaped sets (in fact, drops) with diameter $d - E_\alpha\delta$. Therefore, using Lemma 11 for $(\mathscr{D}^*, K^*, 1/E_\alpha, d - E_\alpha\delta, E_\alpha\delta)$ as $(\mathscr{H}, A, r, D, \delta)$ and that $\delta \leq \delta/2E_\alpha$, we get

$$\begin{aligned} |S(K^*, E_\alpha\delta) \setminus K^*| &\leq C(n, 1/E_\alpha) \frac{E_\alpha\delta}{d - E_\alpha\delta} |K^*| \\ &\leq C(n,1/E_\alpha) \frac{E_\alpha\delta}{d/2} |K| = C\delta|K| , \end{aligned}$$

where $C = C(n, 1/E_\alpha)\frac{E_\alpha}{d/2}$ depends only on n, d and α. This completes the proof of Lemma 16. □

Proof of Theorem 3. Let N be a positive integer which will be defined later. By homogeneity, we can assume that each set of $\mathscr{R}$ has diameter at most 1/2. Let $\mathscr{R}_k$ $(k = 1, 2, \ldots)$ denote the family of sets of $\mathscr{R}$ with diameter between $1/2^{k+1}$ and $1/2^k$, and let

$$\mathscr{H}^j = \mathscr{R}_j \cup \mathscr{R}_{N+j} \cup \mathscr{R}_{2N+j} \cup \ldots \quad (j = 1, \ldots, N) .$$

Clearly $\mathscr{R} = \mathscr{H}^1 \cup \ldots \cup \mathscr{H}^N$.

Fix j. Let $\mathscr{K}_1 = \mathscr{R}_j$. If $\mathscr{K}_1, \ldots, \mathscr{K}_l$ is already defined then let $\mathscr{K}_{l+1}$ be the family of those sets of $\mathscr{R}_{Nl+j}$ which intersect no set of $\mathscr{K}_1, \ldots, \mathscr{K}_l$. Then the diameters of the sets of $\mathscr{K}_l$ are between $1/2^{N(l-1)+j+1}$ and $1/2^{N(l-1)+j}$ $(l = 1, 2, \ldots)$. Let $K_l = \bigcup \mathscr{K}_l$ and $\delta_l = 1/2^{Nl+j}$.

We claim that

$$\bigcup \mathscr{H}^j \subset \bigcup_{l=1}^{\infty} S(K_l, \delta_l)\,. \tag{3}$$

Indeed, if $x \in \bigcup \mathscr{H}^j \setminus \bigcup_{l=1}^{\infty} K_l$ then for an index i we have $x \in R \in \mathscr{R}_{Ni+j}$. On the other hand R cannot be contained in K_{i+1}, so there must be an $l \leq i$ for which R intersects K_l. Since $R \in \mathscr{R}_{Ni+j}$ we have $\operatorname{diam} R \leq 1/2^{Ni+j} \leq \delta_l$. Thus $x \in S(K_l, \delta_l)$, which completes the proof of (3).

If we choose N such that $1/2^{N-1} \leq d/2E_\alpha$ then we can apply Lemma 16 for $\mathscr{K} = \mathscr{K}_l$, $\Delta = 1/2^{N(l-1)+j+1}$, $\delta = 1/2^{N-1}$ to get $K_1^l, \ldots, K_{m_l}^l$ such that this sequence can be distributed in $M(1/2^{N-1})$ families of disjoint sets and

$$|S(K_l, \delta_l) \setminus \bigcup_{i=1}^{m_l} K_i^{m_l}| \leq C \frac{1}{2^{N-1}} |K_l|\,,$$

where C depends only on n, d and α.

Since the sets of $\mathscr{K}_l$ do not intersect the sets of $\mathscr{K}_{l'}$ (if $l \neq l'$), the sets $\{K_i^l : l \in N, i = 1, \ldots, m_l\}$ can also be distributed in $M(1/2^{N-1})$ families of disjoint sets. On the other hand, we have

$$\begin{aligned}
|\bigcup \mathscr{H}_j \setminus \bigcup_{i,l} K_i^l| &\leq |\bigcup_{l=1}^{\infty} S(K_l, \delta_l) \setminus \bigcup_{i,l} K_i^l| \leq |\bigcup_{l=1}^{\infty} (S(K_l, \delta_l) \setminus \bigcup_{i=1}^{m_l} K_i^l)| \\
&\leq \sum_{l=1}^{\infty} |(S(K_l, \delta_l) \setminus \bigcup_{i=1}^{m_l} K_i^l| \leq C \frac{1}{2^{N-1}} \sum_{l=1}^{\infty} |K_l| \\
&\leq \frac{C}{2^{N-1}} |\bigcup \mathscr{H}^j| \leq \frac{C}{2^{N-1}} |\bigcup \mathscr{R}|\,.
\end{aligned}$$

Until this moment j was fixed. Now let $R_1, R_2, \ldots$ be the union of the families $\{K_i^l\}$ we get for $j = 1, \ldots, N$. Then these sets can be distributed in $NM(1/2^{N-1})$ families of disjoint sets and

$$|\bigcup \mathscr{R} \setminus \bigcup_k R_k| \leq \frac{NC}{2^{N-1}} |\bigcup \mathscr{R}|\,.$$

Therefore, if N is an integer such that $\frac{NC}{2^{N-1}} < \varepsilon$ and $1/2^{N-1} \leq d/2E_\alpha$ (depending only on n, d, α and ε) and $M = NM(1/2^{N-1})$ (depending also only on n, d, α and ε), then (i) and (ii) of Theorem 3 are satisfied if m is large enough. $\square$

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