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The Schauder Fixed Point Theorem

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Some results of the type of the Schauder fixed point theorem are presented where the assumptions of compactness and local convexity are omitted. A dual conception of the Kuratowski measure of noncompactness is introduced.

In [2] the author has introduced the notion of topological simplicial space and has proved a version of the Schauder fixed point theorem for some subclass of these spaces.

In this paper we want to construct a tool to estimate approximative fixed points. Our aim will be reached by constructing two dual sequences of functions describing measure of compactness and local convexity.

We shall use notation $[p_0, \ldots, p_n]$ for n-dimensional geometric simplex spanned by vertices p_i, where the points $p_0, \ldots, p_n$ are affinely independent. Each point $x \in [p_0, \ldots, p_n]$, $x = \sum t_i \cdot p_i$, $\sum t_i = 1$, $t_i \geqq 0$, is uniquely determined by its barycentric coordinates t_i. Any continuous map $\sigma : [p_0, \ldots, p_n] \to X$ into topological space X is said to be a *singular simplex* contained in X; and let us introduced the following notations:

$$\mathit{dom}\,\sigma := [p_0, \ldots, p_n], \quad \mathit{im}\,\sigma := \sigma[p_0, \ldots, p_n], \quad \mathit{vert}\,\sigma := \{\sigma(p_0), \ldots, \sigma(p_n)\}$$

The following lemma can be obtained from the Brouwer fixed point theorem.

Lemma on indexed covering. *Let $\{U_0, \ldots, U_n\}$ be an open covering of a topological space and $\sigma : [p_0, \ldots, p_n] \to X$ a singular simplex. Then there exists a sequence $0 \leqq i_0 < \ldots < i_k \leqq n$ of indexes such that $\sigma[p_{i_0}, \ldots, p_{i_k}] \cap U_{i_0} \cap \ldots \cap U_{i_k} \neq \emptyset$*

Proof. Let us put $P := [p_0, \ldots, p_n]$ and $A_i := \sigma^{-1}(U_i)$ for $i = 0, \ldots, n$. The sets A_i are open in P. Define a continuous map $f : P \to P$;

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$$f(x) = \sum_{i=0}^{n} \frac{d_i(x)}{d(x)} \cdot p_i \text{ where } d_i(x) := inf\{\|x - y\| : y \in P \setminus A_i\} \text{ and } d(x) = \sum_{i=0}^{n} d_i(x)$$

Since the sets A_i form an open covering of the simplex P, we infer that $d(x) > 0$ for each point $x \in P$. According to the Brouwer fixed point theorem there exists a point $a \in P$ such that $f(a) = a$. This means that

$$d_i(a) = t_i(a) \,.\, d(a) \quad \text{for each} \quad i = 0, \ldots, n$$

Since the sets A_i are open and $d(a) > 0$ we infer that

$$t_i(a) > 0 \text{ if and only if } a \in A_i \quad \text{for each} \quad i = 0, \ldots, n.$$

Now, let us put $\{i_0, \ldots, i_k\} = \{i \leqq n : t_i(a) > 0\}$. Then, from the above we get

$$a \in [p_{i_0}, \ldots, p_{i_k}] \cap A_{i_o} \cap \ldots \cap A_{i_k}.$$

This completes the proof. □

Recall the definition from [2] of a topological simplicial space. For a given topological space $(X, \mathscr{T})$ denote by Σ the family of all singular simplices contained in X.

A family $\mathscr{F} \subset \Sigma$ is said to be *simplicial structure* in a space X if for each singular simplex $\sigma \in \mathscr{F}$, $\sigma : [p_0, \ldots, p_n] \to X$ and for each sequence of indexes $0 \leqq i_0 < \ldots < i_k \leqq n$ we have $\sigma | [p_{i_0}, \ldots, p_{i_k}] \in \mathscr{F}$.

A triple $(X, \mathscr{T}, \mathscr{F})$, where $\mathscr{T}$ is a topology on X and $\mathscr{F}$ is a simplicial structure in the space $(X, \mathscr{T})$ is said to be *topological simplicial space.* In the case when (X, ϱ) is a metric space or $(X, \|\cdot\|)$ is normed space, the triples $(X, \varrho, \mathscr{F})$, $(X, \|\cdot\|, \mathscr{F})$ will be called metric, or normed simplicial space.

A topological simplicial space $(X, \mathscr{T}, \mathscr{F})$ is said to be *convex* if for each finite set $A \subset X$ there exists a simplex $\sigma \in \mathscr{F}$ such that $A = vert\, \sigma$, and it is *locally convex at a point* $x \in C$ if for each its open neighbourhood U_x there exists an open set V_x, $x \in V_x \subset U_x$ such that

(a) for each finite subset $F \subset V_x$ there exists $\sigma \subset \mathscr{F}$ with $vert\, \sigma = F$, and

(b) for each $\sigma \in \mathscr{F}$; $vert\, \sigma \cap V_x \Rightarrow im\, \sigma \subset U_x$

A simplicial space X which is locally convex at each point $x \in X$ is said to be *locally convex.*

A subsdet $C \subset X$ is said to be *convex* if the conditions (a) and (b) holds (where $C = V_x = U_x$).

Let us recall that a subset $C \subset X$ of a topological linear space X is convex if for each $n + 1$ points $c_0, \ldots, c_n \in C$, each convex combination $\sum_{i=0}^{n} t_i \,.\, c_i$ belongs to C. In our terminology it means that for each singular linear simplex $\sigma \in \mathscr{L}$; *vert* $\sigma \subset C$ implies *im* $\sigma \subset C$. Thus in the case when X is a topological linear space and $\mathscr{F} = \mathscr{L}$ is a simplicial structure consisting of the all linear simplices, then the notion of convexity in our sense coincides with the notion of convexity in the classical sense.

A very important example of simplifical structure is the family $\mathscr{L} \subset \Sigma$ all linear maps (called to be linear simplices), $l: [p_0, \ldots, p_n] \to X$; $l(\sum_{i=0}^n t_i \cdot p_i) = \sum_{i=0}^n t_i \cdot l(p_i)$, where $(X, \mathscr{T})$ is a convex subspace of a linear topological space E. In this case the triple $(X, \mathscr{T}, \mathscr{L})$ is said to be a *linear simplicial space.*

In this remaining part of this paper we shall deal with metric spaces only. If (X, ρ) is a metric space then $B(x, r) := \{y \in X : \rho(x, y) < r\}$ means a ball.

Let $Y \subset X$ be a bounded subset of a metric space (X, ρ). A function $\phi : N \to [0, \infty)$;

$$\phi(n) := inf(r > 0 : Y \subset B(x_0, r) \cup \ldots \cup B(x_n, r) : x_0, \ldots, x_n \in X\}$$

is said to be a sequence function of compactness for the set Y. In this definition we do not assume that the points x_i are distinct. Therefore we have;

$$0 < \phi(n + 1) \leq \phi(n) \quad \text{for each} \quad n \in N$$

The number $\phi(Y) := \lim_{n \to \infty} \phi(n)$ is said to be the Kuratowski measure of noncompatness of Y.

Remarks.

1. It is easy to see that if $X = Y = [0, 1]$, *then* $\phi(n) = \frac{1}{2(n+1)}$.
2. It is left to the reader to check that $\phi(n) \leq \frac{\sqrt{k}}{2E(\sqrt[k]{n})}$ whenever $X = Y = [0, 1]^k$.
3. The following fact is interesting but easy to prove that for each decreasing sequence $\varepsilon_0 > \varepsilon_1 > \ldots > 0$ of positive reals converging to zero, $\varepsilon_n \to 0$ there exists a compact metric space homeomorphic to the Cantor set such that $\phi(n) = \varepsilon_n$.

Now we shall introduce a notion of a sequence function of local convexity which is in some sense dual to sequence function of compactness.

If Y is a subset of a metric simplicial space $(X, \rho, \mathscr{F})$ then define;

$$\psi(n) := inf\{M \geq 1 : [\text{vert}\, \sigma \subset B(y, r) \,\&\, |\text{vert}\, \sigma| \leq n + 1] \Rightarrow$$
$$[\text{im}\, \sigma \subset B(y, M \cdot r)]; \quad \text{for each} \quad y \in Y, r > 0, \sigma \in \mathscr{F}\}.$$

If for each $n \in N$ the number $\psi(n)$ exists then the function $\psi : N \to R$ is said to be a sequence function of local convexity for the subspace Y.

We shall give an example of a metric linear space which is not locally convex and for which the sequence function of local convexity exists.

Example. Fix $0 < p < 1$. Recall that L_p is defined to be the linear F-metric space of all the Lebesgue measurable functions $f : [0, 1] \to R$ with the F-norm;

$$\| f | := \int_0^1 |f(t)|^p \, \mathrm{d}t < \infty .$$

The metric simplicial space $(L_p, \|\cdot\|, \mathscr{L})$ with the linear simplicial structure is obviously convex but it is not locally convex (cf. [3]). We shall show that L_p possesses the sequence function of local convexity.

One can verify that the function $h: T \to R$, $T := \{(t_0, \ldots, t_n) := \sum_{i=0}^{n} t_i = 1, t_i \geq 0\}$, defined by

$$h(t_0, \ldots, t_n) := \sum_{i=0}^{n} t_i^p, \qquad 0 < p < 1$$

assumes the greatest value equal to $(n+1)^{p-1}$ at the point $t = (1/n+1, \ldots, 1/n+1) \in T$. Therefore, if $\sigma : [p_0, \ldots, p_n] \to L_p$ is a linear singular simplex such that $\sigma(p_0) = x_i$, where $x_0, \ldots, x_n \in B(y, r)$, then for $(t_0, \ldots, t_n) \in T$ we have;

$$\left\| \sum_{i=0}^{n} t_i \cdot x_i - y \right\| \leq \sum_{i=0}^{n} \| t_i(x_i - y) \| \leq r \cdot \sum_{i=0}^{n} t_i^p \leq r \cdot (n+1)^{1-p}.$$

This implies that $\psi(n) \leq (n+1)^{p-1}$. □

Main Theorem. *If $g : X \to X$ is a continuous map from a metric simplicial space $(X, \rho, \mathscr{F})$ into itself, then for each $n \in N$ and $\varepsilon_n > 0$ there exists a point $w_n \in X$ such that*

$$\rho(w_n, g(w_n)) < \phi(n) \cdot \psi(n) + \varepsilon_n,$$

where ϕ, ψ mean respectively, the sequence functions of compactness and local convexity of the set $g(X)$.

Proof. Fix $\varepsilon_n > 0$ and choose, $\delta_n > 0$ satisfying
(1) $(\phi(n) + \delta_n) \cdot (\psi(n) + \delta_n) < \phi(n) \cdot \psi(n) + \varepsilon_n$.
According to the definitions of functions ϕ and ψ there exists a finite set of points $x_0, \ldots, x_n \in X$ and positive reals $r < \phi(n) + \delta_n$ and $M < \psi(n) + \delta$ such that
(2) $g(X) \subset B(x_0, r) \cup \ldots \cup B(x_n, r)$,
and for each $x \in g(X)$ and $\sigma \in \mathscr{F}$
(3) $|\text{vert}\, \sigma| \leq n + 1$ and $\text{vert}\, \sigma \subset B(x, r) \Rightarrow \text{im}\, \sigma \subset B(x, r \cdot M)$.
Applying the lemma on indexed covering to the covering $\{U_0, \ldots, U_n\}$, $U_i := g^{-1}(B(x_i, r)$, and a singular simplex $\sigma : [p_0, \ldots, p_n] \to X$ with $\sigma(p_i) = x_i$ we find a point $w_n \in X$ and a sequence of indexes $0 \leq i_0 < \ldots < i_k \leq n$ such that
(4) $w_n \in \sigma[p_{i_0}, \ldots, p_{i_k}] \cap g^{-1}(B(x_{i_0}, r)) \cap \ldots \cap g^{-1}(B(x_{i_k}))$.
From the above it follows that $\sigma(p_{i_0}), \ldots, \sigma(p_{i_k}) \in B(g(w_n), r)$. In view of (3) and (4) we have; $w_n \in B(g(w_n), M \cdot r)$. Thus we have proved that $\rho(w_n, g(w_n)) < \phi(n) \cdot \psi(n) + \varepsilon_n$.

If we assume that balls $B(x, r)$ are convex then it is clear that $\psi(n) = 1$ for each $n \in N$. Now, using compatness arguments we immediately obtain.

Corollary (The Schauder fixed point theorem). *Let $(X, \rho, \mathscr{F})$ be a metric simplicial convex space such that open balls are convex. Then each continuous map $g : X \to X$ where $\overline{g(X)}$ is compact, has a fixed point.*

In known proofs of the classical Schauder theorem, the assumptions on convexity and local convexity are essential. We are going to present a theorem which gives a partial answer to a question when local convexity is preserved under special classes of maps.

A metric simplicial space $(X, \mathscr{T}, \mathscr{F})$ is said to be strongly locally convex if for each compact convex subset $C \subset X$ and its open neighbourhood U, $C \subset U$, there exists an open set V, $C \subset V \subset U$, such that;

$$\text{vert}\,\sigma \subset V \Rightarrow \text{im}\,\sigma \subset U \quad \text{for each} \quad \sigma \in \mathscr{F}.$$

It is clear that each normed space with the linear structure is strongly locally convex.

A continuous map $f : X \to Y$ from a Hausdorff space X onto a Hausdorff space Y is said to be *perfect* if it is closed and $f^{-1}(y)$ is compact for each $y \in Y$ (cf. Engelking [1], p. 236). And f is said to be *monotonic* if $f^{-1}(y)$ is convex for each $y \in Y$.

In [1] one can find the following theorem

If $f : X \to Y$ is a perfect map then $f^{-1}(Z)$ is compact for each compact subset $Z \subset Y$.

Theorem. *Let $f : X \to Y$ be a perfect and monotonic map between Hausdorff spaces and assume that $(X, \mathscr{F})$ is a convex and strongly locally convex simplicial space. Then Y with the simplicial structure $\mathscr{F}_f := \{f \circ \sigma : \sigma \in \mathscr{F}\}$ is a convex and locally convex simplicial space.*

Proof. Since f is onto it is clear that Y is convex. To prove that $\mathscr{F}_f$ is a locally convex simplicial structure let us fix a point $y \in Y$ and its open neighbourhoud U, $y \in U$. From the assumption that $\mathscr{F}$ is strongly locally convex structure on X it follows that there exists an open set W such that $f^{-1}(y) \subset W \subset f^{-1}(U)$ and moreover, the following condition holds;

$$\text{vert}\,\sigma \subset W \Rightarrow \text{im}\,\sigma \subset f^{-1}(U), \quad \text{for each} \quad \sigma \in \mathscr{F}.$$

Since f is closed hence there exists an open neighbourhood V of the point y such that; $y \in V \in U$ and $f^{-1}(V) \subset W$. One can verify, that V satisfies the condition of local convexity;

$$\text{vert}\,(f \circ \sigma) \subset V \Rightarrow \text{im}\,(f \circ \sigma) \subset U, \quad \text{for each}\ \sigma \in \mathscr{F}$$

which completes the proof.

Another kind of theorem on preserving of local convexity is given in [2].

Remark. Observe that the assumption of strong local convexity is essential. To see this, consider the quotient map $f : Q \to Q/\partial Q$, where $Q := \{x \in R^n : \|x\| \leq 1\}$. The quotien space $Q/\partial Q$ is homeomorphic to the sphere $S^n := \{x \in R^{n+1} : \|x\| = 1\}$, which has no fixed point property. Thus, in view of the Schauder fixed point theorem, the simplicial structure $\mathscr{L}_f$, where $\mathscr{L}$ is the linear simplicial structure on Q, cannot be locally convex. It is obvious that $\mathscr{L}_f$ is a convex structure.

References

[1] Engelking, R., *General Topology*, PWN Warszawa 1977.
[2] Kulpa, W., *Topological simplicial spaces*, preprint, pp. 1 – 14.
[3] Rudin, W., *Functional Analysis*, McGraw-Hill Book Company, New York 1973.