# On a Conjecture of L. Veselý

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The aim of this note is to give a negative answer to a question raised by L. Veselý during the Winter School held in Benešova Hora in January 1996. We show that there exists a separable Banach space $Z$ such that for any point $z$ in the unit sphere there is some linear functional which strongly exposes the unit ball at $z$, but on which there is a linear functional which exposes the unit ball without exposing it strongly.

We define functions $\gamma$, $p$ and $q$ on the convex domain $Q = \{(u,v) \in \mathbb{R}^2 : u > |v|\}$ by letting

$$\gamma(u, v) = (u^2 - v^2)^{1/3}$$

$$p(u, v) = \frac{2u}{3(u^2 - v^2)^{2/3}}$$

$$q(u, v) = \frac{-2v}{3(u^2 - v^2)^{2/3}}$$

**Lemma 1.** *If* $(u, v)$ *and* $(u + u', v + v')$ *belong to* $Q$ *and if* $\max(|u'|, |v'|) \leq \frac{1}{3}(u - |v|)$, *we have*

$$\gamma(u + u', v + v') \leq \gamma(u, v) + p(u, v) \, . \, u' + q(u, v) \, . \, v' - \frac{(u - |v|)^{1/3}}{120u^{5/3}} (u'^2 + v'^2) \, .$$

*In particular,* $\gamma$ *is concave on* $Q$.

First of all let us remark that $p(u, v) = \dfrac{\partial \gamma}{\partial u}$ and $q(u, v) = \dfrac{\partial \gamma}{\partial v}$. We have, by Taylor's formula

$$\gamma(u + u', v + v') = \gamma(u, v) + p(u, v) \, . \, v' + q(u, v) \, . \, v' + \int_0^1 (1-\theta) \, D_2(\theta) \, \mathrm{d}\theta$$

where we denote

$$D_2(\theta) = u'^2 r(u + \theta u', v + \theta v') + 2u'v's(u + \theta u', v + \theta v') + v'^2 t(u + \theta u', v + \theta v')$$

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and $r$, $s$ and $t$ are the partial derivatives of order 2 of $\gamma$.

It is easily checked that if $a$, $b$ and $c$ satisfy $a > 0$, $c > 0$ and $ac \geq b^2$, we have, for all real $\xi$ and $\eta$

$$a\xi^2 + 2b\xi\eta + c\eta^2 \geq \frac{ac - b^2}{a + c}(\xi^2 + \eta^2).$$

Since

$$r(u, v) = -\frac{2}{9}\frac{u^2 + 3v^2}{(u^2 - v^2)^{5/3}}$$

$$s(u, v) = \frac{8}{9}\frac{uv}{(u^2 - v^2)^{5/3}}$$

$$t(u, v) = -\frac{2}{9}\frac{3u^2 + v^2}{(u^2 - v^2)^{5/3}}$$

one has

$$\left|\frac{rt - s^2}{r + t}\right| = \frac{1}{6}\frac{(u^2 - v^2)^{1/3}}{u^2 + v^2} \geq \frac{1}{12u^2}(u^2 - v^2)^{1/3} = \frac{(u + |v|)^{1/3}}{12u^2}(u - |v|)^{1/3} \geq \frac{(u - |v|)^{1/3}}{12u^{5/3}}.$$

If $|u'| < \frac{1}{3}(u - |v|) \leq \frac{u}{3}$ and $|v'| < \frac{1}{3}(u - |v|)$, we have $(u + u') - |v + v'| > \frac{1}{3}(u - |v|)$ and $u + u' < \frac{4}{3}u$. Thus we have, for all $\theta \in [0, 1]$

$$D_2(\theta) \leq -\frac{1}{3^{1/3}}\left(\frac{3}{4}\right)^{5/3}\frac{(u - |v|)^{1/3}}{12u^{5/3}}(u'^2 + v'^2) \leq \frac{(u - |v|)^{1/3}}{60u^{5/3}}(u'^2 + v'^2)$$

whence we deduce the expected result since $\int_0^1(1 - \theta)\,d\theta = \frac{1}{2}$.

From now on, we shall speak about $\gamma$ also on the closure of $Q$. The function $\gamma$ is continuous and concave on $\bar{Q}$.

Let $X$ and $Y$ be two Banach spaces. Assume that $\varphi$ is a linear functional on $X$ with norm equal to 1 which does not attain its maximum on the unit ball of $X$. Let us define in the product $Z = X \times Y$ some closed symmetric subset $B$, by letting

$$B = \{z = \{x, y\} : \|y\| + |\varphi(x)| \leq 1 \quad \text{and} \quad \|x\| + \|y\| \leq 1 + \gamma(1 - \|y\|, \varphi(x))\}.$$

**Lemma 2.** *The set $B$ is the unit ball of $Z$ for some equivalent norm* $|||.|||$.

In order to see that $B$ is convex, it is sufficient to show that the mapping: $(x, y) \mapsto \gamma(1 - \|y\|, \varphi(x))$ is concave on the convex set $C = \{(x, y) : \|y\| + |\varphi(x) \leq 1\}$. If $(x_1, y_1)$ and $(x_2, y_2)$ belong to $C$, and $0 \leq t \leq 1$, we have, with $(x, y) = t(x_1, y_1) + (1 - t)(x_2, y_2)$,

$$\begin{aligned}\gamma(1 - \|y\|, \varphi(x)) &\geq \gamma(1 - t\|y_1\| - (1 - t)\|y_2\|, \varphi(tx_1 + (1 - t)x_2)) \\ &= \gamma(t(1 - \|y_1\|) + (1 - t)(1 - \|y_2\|), t\varphi(x_1) + (1 - t)\varphi(x_2)) \\ &\geq t\gamma(1 - \|y_1\|, \varphi(x_1)) + (1 - t)\gamma(1 - \|y_2\|, \varphi(x_2)).\end{aligned}$$

Hence $B$ is the unit ball for some norm on $Z$. Moreover, if $\|x\| + \|y\| \leq 1$, we have

$$\|y\| + |\varphi(x)| \leq \|x\| + \|y\| \leq 1$$

thus $(x, y) \in C$ and

$$\|x\| + \|y\| \leq 1 \leq 1 + \gamma(1 - \|y\|, \varphi(x))$$

whence $(x, y) \in B$. Conversely, if $(x, y) \in B$, we have $\|y\| \leq 1 - \|x\| \leq 1$ and

$$\|x\| + \|y\| \leq 1 + \gamma(1 - \|y\|, \varphi(x)) \leq 1 + (1 - \|y\|)^{2/3} \leq 2$$

thus $\frac{1}{2}(\|x\| + \|y\|) \leq |||(x, y)||| \leq \|x\| + \|y\|$, and this proves these norms are equivalent.

**Definition 3.** *We will say a Banach space E has property* $(*)$ *if, for every unit vector* $x \in E$, *there exists some linear functional on E which strongly exposes the unit ball of E at x.*

**Lemma 4.** *Every L.U.R. space has property* $(*)$.

Let $E$ be a L.U.R. space and $x$ a unit vector of $E$. By Hahn–Banach's theorem, there is a linear functional $f$ such that

$$f(x) = 1 = \|f\| .$$

In order to prove that $f$ strongly exposes the unit ball $B$ of $E$ we have only to prove that every sequence $(x_n)$ in $B$ such that $f(x_n) \to 1$ converges to $x$. But we have

$$1 \geq \left\|\frac{x + x_n}{2}\right\| \geq f\left(\frac{x + x_n}{2}\right) = \frac{f(x) + f(x_n)}{2} \to 1$$

hence $\left\|\frac{x + x_n}{2}\right\| \to 1$, and $x_n \to x$ since $E$ is L.U.R.

**Theorem 5.** *If X and Y have property* $(*)$, *Z has property* $(*)$ *too. Nevertheless, for every unit vector y of Y, there is a linear functional on Z which exposes B at* $(0, y)$ *but does not expose B strongly.*

Let $y \in Y$, with $\|y\| = 1$. By hypothesis there is an $\ell_y \in Y^*$ such that $\|\ell_y\| = \ell_y(y) = 1$ and that for every $y'$ in the unit ball of $Y$ $\ell_y(y') = 1 \Rightarrow y' = y$. We then put

$$\Phi(h, k) = \varphi(h) + \ell_y(k) .$$

We have $\Phi(0, y) = \ell_y(y) = 1$, and for every $(h, k) \in B$,

$$\Phi(h, k) \leq |\varphi(h)| + \|k\| \leq 1$$

thus $\|\Phi\| = 1$. Moreover if $(h, k) \in B$, and $\Phi(h, k) = 1$, we have $(h, k) \in C$ thus

$$1 = \Phi(h, k) \leq |\varphi(h)| + \|k\| \leq 1$$

hence $1 - \|k\| = |\Phi(h)|$ and $\gamma(1 - \|k\|, \varphi(h)) = 0$. It follows that

$$1 = \Phi(h, k) \le \varphi(h) + \|k\| \le \|h\| + \|k\| \le 1 + \gamma(1 - \|k\|, \varphi(h)) = 1$$

thus $\varphi(h) = \|h\|$, what implies $\|h\| = 0$ since $\varphi$ does not attain its norm on the unit ball of $X$. Then $\Phi(h, k) = \ell_y(k) = 1$ and $\|k\| = 1$, whence $k = y$. Thus $\Phi$ exposes $B$ at $(0, y)$.

If $(x_n)$ is a sequence in the unit sphere of $X$, such that $\lim_{n\to\infty} \varphi(x_n) = \|\varphi\| = 1$, we have $\Phi(x_n, 0) = \varphi(x_n) \to 1 = \|\Phi\|$, $|\varphi(x_n)| \le 1$, thus $(x_n, 0) \in C$, and $\|x_n\| \le 1 + \gamma(1, \varphi(x_n))$, hence $|||(x_n, 0)||| \le 1$. Moreover $|||(x_n, 0) - (0, y)||| \ge \frac{1}{2}(\|x_n\| + \|y\|) = 1$. This shows that $\Phi$ does not expose $B$ strongly.

Finally if $|||(x, y)||| = 1$, there is a linear functional $f_x$ (resp. $\ell_y$) with norm 1 on $X$ (resp. $Y$), which strongly exposes the unit ball of $X$ (resp. $Y$) at $\frac{x}{\|x\|}$ if $x \neq 0$ (resp. $\frac{y}{\|y\|}$ if $\|y\| \neq 0$). We then put

$$p = p(1 - \|y\|, \varphi(x))$$
$$q = q(1 - \|y\|, \varphi(x))$$
$$L(h, k) = f_x(h) + \ell_y(k) + p \,.\, \ell_y(k) - q \,.\, \varphi(h)\,.$$

We shall show that $L$ attains its maximum on $B$ at $(x, y)$ and strongly exposes $B$. Suppose that $(x + h, y + k) \in B$. We have

$$\|x + h\| + \|y + k\| - \gamma(1 - \|y + k\|, \varphi(x + h)) \le \|x\| + \|y\| - \gamma(1 - \|y\|, \varphi(x)) = 1\,.$$

Since $x = 0$ or $f_x$ attains its maximum on $B$ at $\frac{x}{\|x\|}$, we have

$$\|x + h\| \ge f_x(x + h) = f_x(x) + f_x(h) = \|x\| + f_x(h)$$

and similarly

$$\|y + k\| \ge \|y\| + \ell_y(k)\,.$$

Finally it follows from Lemma 1 that

$$\begin{aligned}\gamma(1 - \|y + k\|, \varphi(x + h)) &\le \gamma(1 - \|y\|, \varphi(x)) + p \,.\, (\|y\| - \|y + k\|) + q \,.\, \varphi(h)\\ &\le \gamma(1 - \|y\|, \varphi(x)) - p \,.\, \ell_y(k) + q \,.\, \varphi(h)\end{aligned}$$

since $p \ge 0$. We deduce from the above inequalities that

$$\begin{aligned}0 &\ge \|x + h\| + \|y + k\| - 1 - \gamma(1 - \|y + k\|, \varphi(h))\\ &\ge f_x(h) + \ell_y(k) + p \,.\, \ell_y(k) - q \,.\, \varphi(h) = L(h, k)\end{aligned}$$

it is $L(x + h, y + k) \le L(x, y)$ for $(x + h, y + k) \in B$, what means that $L$ attains at $(x, y)$ its maximum on $B$.

Now let $((h_n, k_n))_{n\in\mathbb{N}}$ be a sequence of points of $Z$ such that $|||(x + h_n, y + k_n)||| \le 1$ and $L(x + h_n, y + k_n) \to L(x, y)$. The above inequalities show that

$$0 \le \|x + h_n\| - \|x\| - f_x(h_n) \le -L(h_n, k_n) \to 0$$
$$0 \le \|y + k_n\| - \|y\| - \ell_y(k_n) \le -L(h_n, k_n) \to 0$$

$$\gamma(1-\|y+k_n\|,\varphi(x+h_n))-\gamma(1-\|y\|,\varphi(x))-p.\ell_y(k_n)+q.\varphi(h_n)\geq -L(h_n,k_n)\to 0\,.$$

Replacing if necessary the sequence $((h_n, k_n))_{n\in\mathbb{N}}$ by $((\delta h_n, \delta k_n))_{n\in\mathbb{N}}$, we can and do assume that max $(\|h_n\|, \|k_n\| \leq \frac{1}{3}(1 - \|y\| - |\varphi(x)|)$. We then have, using Lemma 1,

$$\gamma(1-\|y+k_n\|,\varphi(x+h_n)) \leq \gamma(1-\|y\|,\varphi(x)) - p\,.\,\ell_y(k) + q\,.\,\varphi(h) - \sigma_n$$

where

$$\sigma_n = \frac{(1-\|y\|-|\varphi(x)|)^{1/3}}{120(1-\|y\|)^{5/3}}\left((\|y+k_n\|-\|y\|)^2+\varphi(h_n)^2\right).$$

We then have

$$-\sigma_n \geq \gamma(1-\|y+k_n\|,\varphi(x+h_n)) - \gamma(1-\|y\|,\varphi(x)) - p\,.\,\ell_y(k_n) + q\,.\,\varphi(h_n)$$
$$\geq -L(h_n,k_n)\to 0\,.$$

Since $\sigma_n \to 0$, we see that $\varphi(h_n)\to 0$ and that $\|y+k_n\| - \|y\| \to 0$. Then $\ell_y(y+k_n)\to\ell_y(y)$ and $\|y+k_n\|\to\|y\|$, thus $\|k_n\|\to 0$, since $\ell_y$ strongly exposes the unit ball of $Y$. We have

$$f_x(h_n) = L(h_n,k_n) - \ell_y(k_n) - p\,.\,\ell_y(k_n) + q\,.\,\varphi(h_n)\to 0$$

thus $f_x(x+h_n)\to f_x(x) = \|x\|$ and

$$\limsup_{n\to\infty}\|x+h_n\| \leq \limsup_{n\to\infty}\left(1-\|y+k_n\| + \gamma(1-\|y+k_n\|,\varphi(x)+\varphi(h_n))\right)$$
$$\leq 1-\|y\|+\gamma(1-\|y\|,\varphi(x)) = \|y\|\,.$$

And this implies that $\|h_n\|\to 0$, since $f_x$ strongly exposes the unit ball of $X$, thus that the sequence $((x+h_n, y+k_n))_{n\in\mathbb{N}}$ converges to $(x, y)$. This shows that the linear functional $L$ strongly exposes $B$ at $(x, y)$, and completes the proof of the theorem.

It is well known that every separable Banach space can be equipped with a L.U.R. norm (see [1] for instance). If $X$ is the space $\ell^1$ equipped with such a norm, $X$ is not reflexive and thus James' theorem proves the existence of a linear functional $\varphi$ on $X$ with norm 1 which does not attain its norm on the unit ball of $X$. Then, taking $Y = \mathbb{R}$, we get by the previous theorem a proof of the following.

**Theorem 6.** *There is a separable Banach space $Z$ isomorphic to $\ell^1$ such that for every point $z$ in the unit sphere there exists a linear functional which strongly exposes the unit ball of $Z$ at $z$ and that there exists some linear functional which exposes the unit ball without exposing it strongly.*

## Reference

[1] Deville R., Godefroy G. and Zizler V., *Smoothness and Renorming in Banach spaces*, Pitman Monographs and Surveys Pure Appl. Math. **64**, Longman Ed. 1993.