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Equality of Coarse Topologies in Inverse Transformations

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In [2] J. R. Choksi and S. Kakutani proved that all the strong operator (of course) topologies induced from $\mathscr{L}(L^p(m))$, $1 \leq p < \infty$, coincide on the group of invertible transformations. We show that it holds in a more general setting of Orlicz spaces.

1. Introduction

Let us denote the Lebesgue measure on the Borel σ-algebra of [0, 1] by m. We call a Borel mesurable function $\tau : [0.1] \to [0, 1]$ by a transformation if it is nonsingular, that is $m(A) = 0$ implies $m(\tau^{-1}(A)) = 0$. Every two functions equal almost everywhere are identified. A transformation is called invertible if its inverse exists and is also a transformation. We denote by G the group of all invertible transformations.

For $1 \leq p < \infty$ every invertible transformation τ induces an isometry $T_\tau^{(p)}$ of $L^p(m)$ by the formula

$$T_\tau^{(p)} f = f \circ \tau^{-1} \omega_{\tau,p}, \text{ where } f \in L^p(m) \text{ and } \omega_{\tau,p} = \left(\frac{d(m\tau^{-1})}{dm}\right)^{1/p}.$$

The group G may be equipped with the strong operator topology Θ_p taken from $\mathscr{L}(L^p(m))$. These topologies coincide for all $1 \leq p < \infty$ as it was proved by J. R. Choksi and S. Kakutani in [2], Th. 8. In [1] the author showed that if an Orlicz function Φ satisfies the Δ'-condition globally then G may be embedded in the set $\mathscr{L}(L^\Phi(m))$ by the formula

$$\tau \to T_\tau^{(\Phi)} f = f \circ \tau^{-1} \omega_{\tau,\Phi},$$

$$\text{where } \tau \in G,\ f \in L^\Phi(m) \text{ and } \omega_{\tau,\Phi} = \Phi^{-1} \circ \frac{d(m\tau^{-1})}{dm}.$$

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The induced topologies Θ_Φ satisfy $\Theta_\Phi \subset \Theta_p$, $1 \leq p < \infty$ ([1], Cor. 3.7). In this paper we show that if there exists $1 < p < \infty$ such that $\Phi^{1/p}(a + b) \leq \Phi^{1/p}(a) + \Phi^{1/p}(b)$ for all $a, b > 0$ then $\Theta_p \subset \Theta_\Phi$, which means that all such topologies Θ_Φ coincide.

2. The Theorem

Theorem. *Let an Orlicz function Φ satisfy the condition Δ' globally and*

$$(*) \text{ there exist } 1 < p < \infty \text{ such that}$$

$$\Phi^{1/p}(a + b) \leq \Phi^{1/p}(a) + \Phi^{1/p}(b) \text{ for all } a, b > 0\,.$$

$$\text{Then } \Theta_\Phi = \Theta_1\,.$$

Proof. It is enough to show that $\Theta_p \subset \Theta_\Phi$ ([1], Cor. 3.7). By [1], Th. 3.8, we only need to show that if

$$N_\Phi(\omega_{\tau,\Phi} - \omega_{\tau_n,\Phi}) \to 0 \text{ then } \|\omega_{\tau_n,p} - \omega_{\tau,p}\|_p \to 0\,,$$

where τ, τ_n are invertible transformations and $n \to \infty$. By $(*)$ and the properties of Φ we obtain

$$|\Phi^{1/p}(c) - \Phi^{1/p}(d)| \leq \Phi^{1/p}(c - d)$$

for all real c, d. Therefore,

$$(\Phi^{1/p}(c) - \Phi^{1/p}(d))^p \leq \Phi(c - d)\,.$$

Putting $c = \omega_{\tau,\Phi}$ and $d = \omega_{\tau_n,\Phi}$ we obtain

$$\|\omega_{\tau_n,p} - \omega_{\tau,p}\|_p^p =$$

$$\int|\omega_{\tau,p} - \omega_{\tau_n,p}|^p \,\mathrm{dm} \leq \int\Phi \circ (\Phi^{-1} \circ ((\omega_{\tau,p})^p) - \Phi^{-1} \circ ((\omega_{\tau_n,p})^p)\,\mathrm{dm} =$$

$$\int\Phi \circ (\omega_{\tau,\Phi} - \omega_{\tau_n,\Phi})\,\mathrm{dm} \to 0$$

when $n \to \infty$. □

Remarks. 1. Functions x^p, where $1 < p < \infty$, satisfy the condition $(*)$.

2. If for an Orlicz function Φ, which satisfies the Δ'-*condition* globally, there exists $1 < p < \infty$ such that $\Phi^{1/p}$ is concave on $[0, \infty)$ then Φ satisfies $(*)$.

3. The function $\Phi(x) = x^4(|\ln|x|| + 1)$ satisfies the condition Δ' globally and Φ is not equivalent to any function x^p, $1 < p < \infty$. Therefore, $L^\Phi(m)$ is isomorphic to none of the spaces $L^p(m)$. Moreover, $\Phi^{1/6}$ is concave on $[0, \infty)$ as it may be proved by calculating the second derivative Φ''. Thus our theorem is a real generalization of the theorem of J. R. Choksi and S. Kakutani.

4. It was noticed by Professor H. Hudzik that if $\Phi(x) = x^2$ for $|x| \leq 1$ and $\Phi(x) = |x^3|$ for $|x| > 1$ then $\Phi^{1/p}$ is not concave on $[0, \infty)$ for any $1 < p < \infty$ although Φ satisfies the Δ'-*condition* globally. However, it easy to check that Φ satisfies $(*)$ with $p = 3$.

5. Professor H. Hudzik proved also that for each Orlicz function Φ which satisfies the condition Δ' globally there exists $1 < p < \infty$ and a concave (on $[0, \infty)$) function Ψ which is equivalent to $\Phi^{1/p}$, that is $\Psi \sim \Phi^{1/p}$. The author does not know for now whether this improves his theorem.

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