On the Suslin Number of Subgroups of Products of Countable Groups

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We construct a subgroup of $\mathbb{Z}^{\mathfrak{c}}$, where $\mathbb{Z}$ is the group of integers, which does not have the Suslin property. This answers a question of A. V. Arhangel'skiĭ.

The *Suslin number* $c(X)$ of a topological space X is defined as folow: if κ is a cardinal, then $c(X) \le \kappa$ if and only if any family of pairwise disjoint non-empty open sets in X has cardinality $\le \kappa$. We denote by $\mathbb{Z}$ the discrete group of integers.

If G is topological group and $c(G) = \kappa$, then for every subgroup H of G we have $c(H) \le 2^{\kappa}$, and this bound can be attained [U1]. In particular, if G is a subgroup of the product of countable groups, then $c(G) \le \mathfrak{c} = 2^{\omega}$. There exists a subgroup G of $A^{\mathfrak{c}}$, where A is the discrete free abelian group on a countable set, such that $c(G) = \mathfrak{c}$ [U2]. A. V. Arhangel'skiĭ asked if there exists a subgroup G of $\mathbb{Z}^{\mathfrak{c}}$ such that $c(G) = \mathfrak{c}$. The aim of this note is to answer this question in the positive.

Example. *There exists a subgroup G of $\mathbb{Z}^{\mathfrak{c}}$ such that $c(G) = \mathfrak{c}$.*

The proof is based on the following lemma:

Lemma. *Let I be a set of cardinality $\mathfrak{c}$. There exists an $I \times I$-matrix (a_{ij}) with integer coefficients (in other words, a map $I \times I \to \mathbb{Z}$) such that for any distinct $i,j \in I$ there exists a prime p such that $a_{ij} \not\equiv a_{ji} \pmod{p}$ and $a_{ih} \equiv a_{jh} \pmod{p}$ for every $h \in I \setminus \{i,j\}$.*

Proof. We may assume that $I = 2^{\omega}$. Let $T = 2^{<\omega}$ be the set of all finite sequences with values 0 or 1. For any distinct $i,j \in I$ let $n(i,j) \in \omega$ be the smallest integer n such that $i|n \neq j|n$, and set $w(i,j) = i|n \in T$. We shall construct a function $f : T \to \mathbb{Z}$ such that the matrix $(i,j) \mapsto a_{ij} = f(w(i,j))$ has the required property.

We denote by $s\hat{\ }t$ the concatenation of sequences $s, t \in T$. For every $s \in T$ define by induction on the length of s an integer $f(s)$ and a prime $p(s)$ so that:

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(1) $f(s\hat{}0) \equiv f(s\hat{}1\hat{}t) (\mathrm{mod}\, p(s))$ and $f(s\hat{}1) \equiv f(s\hat{}0\hat{}t) (\mathrm{mod}\, p(s))$ for every $s \in T$ and every $t \in T \setminus \{\emptyset\}$;
(2) $f(s\hat{}0) \not\equiv f(s\hat{}1) (\mathrm{mod}\, p(s))$;
(3) the primes $p(s)$ are pairwise distinct.

Pick $f(\emptyset) \in \mathbb{Z}$ arbitrarily. Suppose that $f(t)$ and $p(s)$ have been defined for every t of length $\leq n$ and every s of length $< n$. Let $s \in T$ be a sequence of length $n, t_0 = s\hat{}0$, $t_1 = s\hat{}1$. We must define $p(s)$, $f(t_0)$, $f(t_1)$. For every $k < n$ let $s_k = s|k$ be the restriction of s to k, and let $u_k = s_k\hat{}(1 - s(k))$ be the sequence of length $k + 1$ such that $u(i) = s(i)$ for all $i < k$ and $u(k) \neq s(k)$. The system of n congruences

$$x \equiv f(u_k) (\mathrm{mod}\, p(s_k)), \quad k = 0, \ldots, n - 1$$

has an integer solution, since the primes $p(s_k)$ are pairwise distinct. Let $f(t_0)$ and $f(t_1)$ be any two distinct solutions of this system. Let $p(s)$ be any prime which is not a divisor of $f(t_0) - f(t_1)$ and which is distinct from all the primes of this form which have already been defined.

The functions f and p are constructed. The properties (2) and (3) obviously hold. Let us check the property (1). Let $c = a\hat{}0$, $d = a\hat{}1\hat{}b$ for some $a, b \in T$, $b \neq \emptyset$. Denote by $l(u)$ the length of a sequence $u \in T$. If $l(d) = n + 1$, $l(a) = k$ and $s = d|n$, then in the notation of the preceding paragraph we have $d = s\hat{}0$ or $d = s\hat{}1$, $a = s_k$ and $c = u_k$. Thus the congruence $f(d) \equiv f(u_k) (\mathrm{mod}\, p(s_k))$, which holds by the construction, can be rewritten in the form $f(d) \equiv f(c) (\mathrm{mod}\, p(a))$, and this is the property (1). The case $c = a\hat{}1$, $d = a\hat{}0\hat{}b$ is similar.

We show that the $I \times I$-matrix (a_{ij}), where $a_{ij} = f(w(i,j))$, has the property required by the lemma. Let i and j be distinct elements of $I = 2^\omega$. Let $k = n(i,j) - 1$ be the greatest integer r such that $i|r = j|r$, and let $s = i|k = j|k$. Let $p = p(s)$. Since $\{w(i,j), w(j,i)\} = \{s\hat{}0, s\hat{}1\}$, the property (2) above implies that $a_{ij} \not\equiv a_{ji} (\mathrm{mod}\, p)$. Let us show that $a_{ih} \equiv a_{jh} (\mathrm{mod}\, p)$ for every $h \in I \setminus \{i,j\}$. If $n(i,h) < n(i,j)$, then $w(i,h) = w(j,h)$ and hence $a_{ih} = a_{jh}$. If $n(i,h) = n(i,j)$, then $n(j,h) > n(i,j)$ and we have $w(i,h) = s\hat{}\varepsilon$, $w(j,h) = s\hat{}(1 - \varepsilon)\hat{}t$ for some $s, t \in T$, $\varepsilon \in \{0,1\}$, $t \neq \emptyset$. Similarly, if $n(i,h) > n(i,j)$, then $w(j,h) = s\hat{}\varepsilon$, $w(i,h) = s\hat{}(1 - \varepsilon)\hat{}t$. The property (1) implies that $a_{ih} \equiv a_{jh} (\mathrm{mod}\, p)$. □

We now construct our example. Let A be a discrete free abelian group of rank 2, and let $\{v, w\}$ be a basis of A. Let I be a set of cardinality $\mathfrak{c}$. The topological group A^I is isomorphic to $\mathbb{Z}^\mathfrak{c}$. Let (a_{ij}) be an ineger $(I \times I)$-matrix satisfying the condition of the Lemma. Define an $(I \times I)$-matrix $B = (b_{ij})$ with coefficients in the group A by $b_{ij} = a_{ij}v$ if $i \neq j$ and $b_{ii} = w$ for every $i \in I$. Let G be the subgroup of A^I generated by the columns of the matrix B. We claim that $c(G) = \mathfrak{c}$.

For every $i \in I$ let $U_i = \{f \in A^I : f(i) = w\}$, and let $c_i \in A^I$ be the ith column of B, considered as the function $j \mapsto b_{ji}$. Set $V_i = U_i \cap G$. Each V_i is non-empty, since $c_i \in V_i$. We claim that the family $\{V_i : i \in I\}$ of open sets in G is disjoint.

Let $i, j \in I$ be distinct. To prove that $V_i \cap V_j = \emptyset$, we must show that there is no $f \in G$ such that $f(i) = f(j) = w$. Assume the contrary. Let $f \in G$ be such that $f(i) = f(j) = w$. There exists a family $\{n_h : h \in I\}$ of integers such that $n_h = 0$ for all but finitely many $h \in I$ and $f = \sum n_l c_h$. Since $c_h(i) = a_{ih} v$ for every $h \neq i$ and $c_i(i) = w$, we have

$$w = f(i) = n_i c_i(i) + \sum_{h \neq i} n_h c_h(i) = n_i w + \Big(\sum_{h \neq i} n_h a_{ih}\Big) v. \tag{A}$$

It follows that $n_i = 1$. Similarly, $n_j = 1$. Set $H = I \setminus \{i,j\}$. Now (A) implies that

$$a_{ij} + \sum_{h \in H} n_h a_{ih} = 0; \tag{B}$$

similarly, we have

$$a_{ji} + \sum_{h \in H} n_h a_{jh} = 0. \tag{C}$$

Subtract (B) from (C). We obtain

$$a_{ji} - a_{ij} = \sum_{h \in H} n_h (a_{ih} - a_{jh}). \tag{D}$$

Since the matrix (a_{ij}) satisfies the condition of the Lemma, there exists a prime p which divides $a_{ih} - a_{jh}$ for every $h \in H$ and does not divide $a_{ji} - a_{ij}$. This contradicts (D).

We have thus proved that the sets V_i are pairwise disjoint. Hence $c(G) \geq \mathfrak{c}$. The reverse inequality is obvious. □

Let A be a free abelian group. Let $\mathscr{F}$ be the family of all subgroups $H \subset A$ such that the quotient group A/H is finitely generated. Equip A with the group topology $\mathscr{T}$ for which $\mathscr{F}$ is a basis at 0. The topology $\mathscr{T}$ is the weakest group topology on A for which every homomorphism $A \to \mathbb{Z}$ is continuous. Our main result can be reformulated as follows: the Suslin number of the topological group A equals $\max(|A|, \mathfrak{c})$. Indeed, A embeds as a topological subgroup in a power of $\mathbb{Z}$, hence $c(A) \leq \mathfrak{c}$ according to [U1]. On the other hand, if $|A| \geq \mathfrak{c}$, then A admits a continuous homomorphism onto the group G constructed in the Example above, and hence $c(A) \geq c(G) = \mathfrak{c}$.

References

[U1] Uspenskij V. V., *The topological group generated by a Lindelöf Σ-space has the Suslin property,* Dokl. AN SSSR **265** (1982), 823–826; English transl. in: Soviet Math. Dokl. **26** (1982), 166–169.

[U2] Uspenskij V. V., *On the Souslin number of topological groups and their subgroups,* Leningrad Internat. Topology Conf. (1982), Abstracts, Nauka, Leningrad, 1982, p. 162.