B. de Pagter; Anton Roelof Schep

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Positive Definite Diagonal Sequences

B. DE PAGTER AND A. R. SCHEP

Delft, Columbia*)

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Introduction

In the first section of the paper we develop a systematic theory of positive definite sequences in complex Banach lattices. It turns out that the classical theory (Carathéodory, Herglotz, Toeplitz) has a natural extension to the setting of general Banach lattices. In the second section of the paper we discuss how to generate positive definite sequences of operators in $\mathscr{L}_r(E)$, the space of all regular operators on the Banach lattice E. In particular, it is shown that for any operator $T \in \mathscr{L}_r(E)$ with $r(|T|) = 1$, the sequence $\{D_n\}_{n\in\mathbb{Z}}$ of diagonal operators, defined by $D_n = \mathscr{D}(T^n)$ for $n \geq 0$ and $D_n = \overline{\mathscr{D}(T^{-n})}$ for $n < 0$, is an operator valued positive definite sequence. Here $\mathscr{D}$ denotes the diagonal projection from the space $\mathscr{L}_r(E)$ onto the center $Z(E)$ of E. As a consequence of the general results of section 1 we then obtain the so-called Andô inequality: if $T \in \mathscr{L}_r(E)$ then $|TR(\lambda, T) \leq |\lambda R(\lambda, T)|$ for all $|\lambda| > r(|T|)$. We thus recover some of the results of [5], where the Andô inequality was proved directly, and the positive definiteness of $\{D_n\}_{n\in\mathbb{Z}}$ was obtained as a consequence of this inequality. The present approach, however, puts these results in a general perspective and gives more information about which positive definite sequences can arise in such a way. Moreover, the present framework allows us to generate positive definite sequences in $\mathscr{L}_r(E)$ by projecting the powers of a regular operator into the principal band generated by an order continuous Riesz homomorphism (or the principal band generated by an interval preserving operator). In case $E = \ell_p(I)$ with $1 \leq p < \infty$ we can associate with each $T \in \mathscr{L}_r(E)$ a matrix (t_{ij}) and then $\mathscr{D}(T)$ will have a diagonal matrix, with t_{ii} on the diagonal. Now the results of section 2 imply that if $r(|T|) = 1$, then the diagonal elements of the matrix of T^n define positive definite scalar sequences. In section 3

*) Department of Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands
Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208, U.S.A.

of the paper we show that the same is true for contractions T on $\ell_p(I)$, where T is no longer assumed to be regular.

1. Positive definite sequences in a Banach lattice

In this section we extend the classical theory of positive sequences to complex Banach lattices. We will assume that the reader is familiar with the basic terminology and theory of Banach lattices, as can be found e.g. in the books [1], [7], [8] and [9]. All Banach lattices E in this paper will be assumed to be *complex Banach lattices*, i.e., $E = \mathrm{Re}\, E \oplus i\, Re\, E$, where $\mathrm{Re}\, E$ is a real Banach lattice. For an element $z = x + iy \in E$ with $x, y \in \mathrm{Re}\, E$ we define $\bar{z}$ as $x - iy$.

Definition 1.1. *Let E be a complex Banach lattice. The sequence $(x_n)_{n\in\mathbb{Z}}$ in E is called positive definite if for all finite sequences λ_n of complex scalars we have $\Sigma_l \Sigma_m \lambda_l \bar{\lambda}_m x_{l-m} \geq 0$.*

We shall first derive some elementary properties of positive definite sequences.

Lemma 1.2. *Let E be a complex Banach lattice and let $(x_n)_{n\in\mathbb{Z}}$ be a positive definite sequence in E. Then the following hold.*

(1) $x_0 \geq 0$
(2) $x_{-n} = \bar{x}_n$ *for all* $n \geq 1$
(3) $|x_n| \leq x_0$ *for all* $n \in \mathbb{Z}$.

Proof. Part (1) is obvious. For (2) take $\lambda_0 = 1$ and $\lambda_k = 0$ for all $k \neq n$. Then $\Sigma_l \Sigma_m \lambda_l \bar{\lambda}_m x_{l-m} = (1 + |\lambda_n|^2) x_0 + \bar{\lambda}_n x_{-n} + \lambda_n x_n \geq 0$ for all $\lambda_n \in \mathbb{C}$. Hence $\mathrm{Im}(\bar{\lambda}_n x_{-n} + \lambda_n x_n) = 0$ for all $\lambda_n \in \mathbb{C}$. Taking $\lambda_n = 1$ we see that $\mathrm{Im}\, x_{-n} = -\mathrm{Im}\, x_n$ and taking $\lambda_n = i$ we get that $\mathrm{Re}\, x_{-n} = \mathrm{Re}\, x_n$, i.e. $x_{-n} = \bar{x}_n$. To prove (3) let λ_0 and λ_k be as in the proof of part (2). Then by (2) we have $(1 + |\lambda_n|^2) x_0 + 2\,\mathrm{Re}(\lambda_n x_n) \geq 0$ for all $\lambda_n \in \mathbb{C}$. Now take $\lambda_n = -e^{i\theta}$ to get $\mathrm{Re}\,(e^{i\theta} x_n) \leq x_0$ for all $\theta \in [0, 2\pi]$. Hence $|x_n| = \sup_\theta \mathrm{Re}(e^{i\theta} x_n) \leq x_0$ for all $n \leq 1$. □

The following proposition establishes the important relation between positive definite sequences and positivity of vector valued trigonometric series.

Proposition 1.3. *Let E be a complex Banach lattice and $(x_n)_{n\in\mathbb{Z}}$ in E be a sequence. Then the following are equivalent.*

(1) *The sequence $(x_n)_{n\in\mathbb{Z}}$ is positive definite.*
(2) *The series $\Sigma_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} x_n$ converges in norm and is ≥ 0 for all $\theta \in [0, 2\pi]$ and all $0 \leq r < 1$.*

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Proof. The proof is similar to the scalar case, or can be deduced from the scalar case by observing that the sequence $(x_n)_{n\in\mathbb{Z}}$ is positive definite in E if and only if the sequnce of scalars $(\phi(x_n))_{n\in\mathbb{Z}}$ is positive definite for all $0 \leq \phi \in E^*$. □

The following theorem can now be considered as the vector valued version of the Herglotz theorem. We denote with $C(\mathbb{T})$ the space of complex valued continuous 2π-periodic functions on $[0, 2\pi]$.

Theorem 1.4. *Let E be a complex Banach lattice and $(x_n)_{n\in\mathbb{Z}}$ in E be a sequence. Then the following are equivalent.*

(1) *The sequence $(x_n)_{n\in\mathbb{Z}}$ is positive definite.*
(2) *There exists a positive linear operator $A : C(\mathbb{T}) \to E$ such that $A(e^{in\theta}) = x_n$ for all $n \in \mathbb{Z}$.*

Proof. Assume (1) holds. Then by Proposition 1.3 we have that $\Sigma_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} x_n$ converges in norm and is ≥ 0 for all $\theta \in [0, 2\pi]$ and all $0 \leq r < 1$. Therefore we can define for $0 < r < 1$ the positive linear operator $A_r : C(\mathbb{T}) \to E$ by means of

$$A_r(f) = \frac{1}{2\pi} \int_0^{2\pi} \Big(\sum_{n\in\mathbb{Z}} r^{|n|} e^{-in\theta} x_n \Big) f(\theta) d\theta$$
$$= \sum_{n\in\mathbb{Z}} r^{|n|} \hat{f}(n) x_n .$$

Now $A_r(\mathbf{1}) = x_0$ for all $0 \leq r < 1$, so that $\|A_r\| = \|x_0\|$ for all $0 \leq r < 1$. Now it is obvious that $\lim_{r\uparrow 1} A_r(p)$ exists in norm in E for every trigonometric polynomial p. Since A_r is uniformly bounded and the trigonometric polynomials are dense in $C(\mathbb{T})$, it follows that $A(f) = \lim_{r\uparrow 1} A_r(f)$ exists for all $f \in C(\mathbb{T})$. Clearly $A \geq 0$ and $A(e^{in\theta}) = x_n$ for all n.

Now assume (2) holds. Let (λ_n) be a finite sequence in $\mathbb{C}$. Then

$$\sum_l \sum_m \lambda_l \bar{\lambda}_m x_{l-m} = \sum_l \sum_m \lambda_l \bar{\lambda}_m A(e^{i(k-l)\theta}) = A\Big(\Big| \sum_k \lambda_k e^{ik\theta} \Big|^2 \Big) \geq 0 ,$$

so (1) holds. □

The following corollary can be viewed as an analogue of the von Neumann inequality for positive definite sequences.

Corollary 1.5. *Let E be a complex Banach lattice and $(x_n)_{n\in\mathbb{Z}}$ in E be a positive definite sequence. Let $f(z) = \sum_{n\geq 0}^N a_n z^n$ be a complex polynomial. Then we have*

$$\Big\| \sum_{n\leq 0}^{N} a_n x_n \Big\| \leq \Big(\sup_{|z|=1} |f(z)| \Big) \|x_0\| .$$

Proof. Immediate from the above theorem, if we observe that $g(\theta) = f(e^{i\theta})$ satisfies $A(g) = \sum_{n\geq 0}^N a_n x_n$. □

One can extend the above corollary as follows to functions in the disc algebra. Let $f(z) = \sum_{n\geq 0} a_n z^n$ define a continuous function on $|z| \leq 1$, which is analytic on $|z| < 1$. Then for $0 < r < 1$ the series $f_r(z) = \sum_{n\geq 0} a_n r^n z^n$ converges uniformly on $|z| = 1$, so that $A(\sum_{n\geq 0} a_n r^n z^n) = \sum_{n\geq 0} a_n r^n x_n$ converges in norm in E. From this one can derive easily that the Abel sum $(A)\sum_{n\geq 0} a_n x_n = \lim_{r\uparrow 1} \sum_{n\geq 0} a_n r^n x_n$ exists in E. Similarly as in the corollary one proves now that $\|(A)\sum_{n\geq 0} a_n x_n\| \leq$ $\leq (\sup_{|z|=1} |f(z)|)\|x_0\|$.

We now present the Carathéodory-Herglotz-Toeplitz characterization of positive definite sequences. We first introduce a notation. For a sequence $(x_n)_{n\geq 0}$ in a complex Banach lattice E we define the sequence $(\hat{x}_n)_{n\in\mathbb{Z}}$ by means of $\hat{x}_n = x_n$ for $n \geq 0$ and $\hat{x}_n = \bar{x}_{-n}$ for $n < 0$.

Theorem 1.6. *Let E be a complex Banach lattice and $(x_n)_{n\geq 0}$ be a norm bounded sequence. Then $(\hat{x}_n)_{n\in\mathbb{Z}}$ is positive definite if and only if* $\mathrm{Re}(\sum_{n\geq 0} x_n z^n) \geq$ $\geq \frac{1}{2}x_0$ *for all* $|z| < 1$.

Proof. Assume first $(\hat{x}_n)_{n\in\mathbb{Z}}$ is a positive definite sequence. Let $A : C(\mathbb{T}) \to E$ be the positive linear operator of Theorem 1.4 such that $A(e^{in\theta}) = x_n$. Then $\sum_{n\geq 0} x_n z^n = A(\sum_{n\geq 0} e^{in\theta} z^n) = A(\frac{1}{1-ze^{i\theta}})$. For $|z| < 1$ we have the inequality

$$\mathrm{Re}\left(\frac{1}{1 - ze^{i\theta}} - \frac{1}{2}\right) = \frac{1}{2}\,\mathrm{Re}\,\frac{1 - |z|^2}{|1 - ze^{i\theta}|^2} \geq 0\,.$$

Hence for all $|z| < 1$ we have

$$\mathrm{Re}\left(\sum_{n\geq 0} x_n z^n\right) = A\left(\mathrm{Re}\left(\frac{1}{1 - ze^{i\theta}}\right)\right) \geq \frac{1}{2}A(1) = \frac{1}{2}x_0\,.$$

For the converse implication, if $\mathrm{Re}(\sum_{n\geq 0} x_n z^n) \geq \frac{1}{2}x_0$ for all $|z| < 1$, then $\sum_{n\in\mathbb{Z}} \hat{x}_n r^{|n|} e^{i\theta n} = 2\,\mathrm{Re}\{(\sum_{n\geq 0} x_n r^n e^{i\theta n}) - \frac{x_0}{2}\} \geq 0$. From Proposition 1.3 it follows that the sequence $(\hat{x}_n)_{n\in\mathbb{Z}}$ is positive definite. □

Corollary 1.7. *Let E be a complex Banach lattice and $(x_n)_{n\geq 0}$ be a sequence such that the sequence $(\hat{x}_n)_{n\in\mathbb{Z}}$ is positive definite. Then $|\sum_{n\geq 1} x_n z^n| \leq |\sum_{n\geq 0} x_n z^n|$ for all $|z| < 1$. Conversely, if $(x_n)_{n\geq 0}$ is a norm bounded sequence with $x_0 \geq 0$, $|x_n| \in \{x_0\}^{dd}$ for $n \geq 1$ and such that $|\sum_{n\geq 1} x_n z^n| \leq |\sum_{n\geq 0} x_n z^n|$ for all $|z| < 1$, then $(\hat{x}_z)_{n\in\mathbb{Z}}$ is positive definite.*

Proof. Let $|z| < 1$ and let $y = \sum_{n\geq 0} x_n z^n$. Then under either hypothesis $y \in$ $\in \{x_0\}^{dd}$. Now the inequality $|y - \frac{x_0}{2}| \leq |y|$ is equivalent with $\mathrm{Re}\, y \geq \frac{x_0}{2}$. This can be seen for instance by means of the Kakutani representation of the principal ideal generated by $|y| + x_0$. The corollary therefore follows from the above theorem. □

Now we shall discuss the convergence of the Césaro averages $\frac{1}{N}\sum_{n=0}^{N-1} x_n$. We start with an observation, which allows us to transfer the problem of convergence in E to the corresponding problem in the center $Z(E)$ of E, in case E is Dedekind complete. Recall first some definitions. A complex Banach latice E is called Dedekind complete, if $\operatorname{Re} E$ is a Dedekind complete vector lattice. We denote by $\mathscr{L}_r(E)$ the space of regular operators on E. When E is Dedekind complete, then also $\mathscr{L}_r(E)$ is a Dedekind complete Banach lattice and every $T \in \mathscr{L}_r(E)$ has a modulus $|T|$. The center $Z(E)$ of E can now be defined by $Z(E) = \{T \in \mathscr{L}_r(E) : |T| \leq \lambda I \text{ for some } \lambda \geq 0\}$.

Proposition 1.8. *Let E be a Dedekind complete complex Banach lattice. Then a sequence $(x_n)_{n\in\mathbb{Z}}$ in E is positive definite if and only if $x_0 \geq 0$ and there exists a positive definite sequence $(\pi_n)_{n\in\mathbb{Z}}$ in $Z(E)$ such that $\pi_n x_0 = x_n$ for all $n \in \mathbb{Z}$.*

Proof. If $x_0 \geq 0$ and there exists a positive definite sequence $(\pi_n)_{n\in\mathbb{Z}}$ in $Z(E)$ such that $\pi_n x_0 = x_n$ for all $n \in \mathbb{Z}$, then it is immediate from the definition that $(\pi_n x_0)_{n\in\mathbb{Z}}$ is positive definite in E. Conversely assume that $(x_n)_{n\in\mathbb{Z}}$ is a positive definite sequence in E. Then by Proposition 1.2 we have that $x_0 \geq 0$ and $|x_n| \leq x_0$ for all $n \in \mathbb{Z}$. Since E is Dedekind complete, there exist π_n in $Z(E)$ such that $x_n = \pi_n x_0$ for all $n \in \mathbb{Z}$. We can assume that $\pi_n = 0$ on $\{x_0\}^d$ for all $n \in \mathbb{Z}$. Now for all finite sequences λ_n of a complex scalars we have $(\sum_l \sum_m \lambda_l \bar{\lambda}_m \pi_{l-m}) x_0 \geq 0$. This implies that $\sum_l \sum_m \lambda_l \bar{\lambda}_m \pi_{l-m} \geq 0$ on $\{x_0\}^{dd}$ for all finite sequences λ_n of complex scalars and thus also $\sum_l \sum_m \lambda_l \bar{\lambda}_m \pi_{l-m} \geq 0$ on E for all finite sequences λ_n of complex scalars. Hence $(\pi_n)_{n\in\mathbb{Z}}$ is a positive definite sequence in $Z(E)$. □

Theorem 1.9. *Let E be a complex Dedekind complete Banach lattice and $(x_n)_{n\in\mathbb{Z}}$ in E be a positive definite sequence. Then $\frac{1}{N}\sum_{n=0}^{N-1} x_n$ converges in order to a positive element of E. In particular, when E has order continuous norm, then $\frac{1}{N}\sum_{n=0}^{N-1} x_n$ converges in norm to a positive element of E.*

Proof. Let $(\pi_n)_{n\in\mathbb{Z}}$ in $Z(E)$ be the positive definite sequence in $Z(E)$ such that $\pi_n x_0 = x_n$ for all $n \in \mathbb{Z}$ and let $A : C(\mathbb{T}) \to Z(E)$ be the positive linear operator of Theorem 1.4 such that $A(e^{in\theta}) = \pi_n$. Then $\frac{1}{N}\sum_{n=0}^{N-1} \pi_n = A(\frac{1}{N}\sum_{n=0}^{N-1} e^{in\theta})$. Now $\frac{1}{N}(\sum_{n=0}^{N-1} e^{in\theta})$ converges pointwise to $\chi_{\{0\}}$, so by the Lebesgue Dominated Convergence theorem we get that for all $0 \leq f \in Z(E)^*$ we have $\langle f, \frac{1}{N}\sum_{n=0}^{N-1} x_n\rangle = \langle A^*f, \frac{1}{N}\sum_{n=0}^{N-1} e^{in\theta})\rangle \to \langle A^*f, \chi_{\{0\}}\rangle = \langle f, A^{**}(\chi_{\{0\}})\rangle$ as $N \to \infty$. Observe the here $\chi_{\{0\}}$ is considered an element of $C(\mathbb{T})^{**}$ by defining $\langle \mu, \chi_{\{0\}}\rangle = \mu(\{0\})$.We now will use the fact that $Z(E) \cong C(K)$ and that there exists a σ-order continuous positive projection P from the Borel functions on K to $C(K)$ (see [5] for a more detailed discussion of this fact and see [3, Lemma 5.6.22 and Remark 5.6.24] for a self contained proof of the existence of such a projection). Now taking $f = \delta_\omega$ with $\omega \in K$ we obtain that $\langle \delta_\omega, \frac{1}{N}\sum_{n=0}^{N-1} \pi_n\rangle. \to \langle A^*\delta_\omega, \chi_{\{0\}}\rangle = \langle \delta_\omega, A^{**}(\chi_{\{0\}})\rangle$ as $N \to \infty$. Hence $g(\omega) = \langle A^{**}(\chi_{\{0\}}), \delta_\omega\rangle$ is a Borel function on K such that $\frac{1}{N}\sum_{n=0}^{N-1} \pi_n$ converges pointwise to g on K. Hence $\frac{1}{N}\sum_{n=0}^{N-1} \pi_n$ converges in order to Pg in $Z(E)$. It follows now directly that $\frac{1}{N}\sum_{n=0}^{N-1} x_n$ converges in order to $(Pg)x_0$ in E. □

Remark. Without the assumption of Dedekind completeness one can not expect that for positive definite sequences $(x_n)_{n\in\mathbb{Z}}$ in E the sequence $\frac{1}{N}\sum_{n=0}^{N-1} x_n$ will converge in order to an element in E^+. Take e.g. $x_1(t) = \chi_{[0,\frac{1}{2}]}(t) + (2-2t)\chi_{[\frac{1}{2},1]}(t)$ in $E = C[0,1]$. Then it is easy to verify that $x_n = x_1^{|n|}$ defines a positive definite sequence in E for which the sequence $(\frac{1}{N}\sum_{n=0}^{N-1} x_n)$ does not converge in order to an element in E.

2. Positive definite diagonal sequences

In this section we shall show that certain sequences of operators on a Banach lattice E are positive definite. *We assume throughout this section of the paper that E is Dedekind complete* (i.e., that $\operatorname{Re} E$ is a Dedekind complete vector lattice). We denote by $\mathscr{L}_r(E)$ the space of regular operators on E, i.e., those operators which can be written as linear combinations of positive operators. Note that $\mathscr{L}_r(E)$ coincides with the space of all order bounded operators on E, and is also denoted by $\mathscr{L}_b(E)$. The space $\mathscr{L}_r(E)$ is a subspace of the space $\mathscr{L}(E)$ of all bounded linear operators on E, but in general $\mathscr{L}_r(E)$ is not a closed subspace. Under the assumptions, the space $\mathscr{L}_r(E)$ is a complex vector lattice (where the positive cone consists of all positive operators on E), and for $T\in\mathscr{L}_r(E)$ the modulus $|T|$ is given by $|T|\,(x) = \sup\{|Tz| : |z| \le x\}$ for all $x \ge 0 \in E$. For any $T\in\mathscr{L}_r(E)$, the *regular norm* is defined by $\|T\|_r := \|\,|T|\,\|$, and then $(\mathscr{L}_r(E), \|\cdot\|_r)$ is a complex Banch lattice algebra. Note that $\|T\| \le \|T\|_r$ holds for all $T\in\mathscr{L}_r(E)$, but in general the two norms $\|\cdot\|$ and $\|\cdot\|_r$ are not equivalent. Recall that the center $Z(E)$ of E is defined by $Z(E) = \{T\in\mathscr{L}_r(E) : |T| \le \lambda I \text{ for some } \lambda \ge 0\}$. It is well known that $Z(E)$ is a commutative full subalgebra of the space $\mathscr{L}(E)$. If $T\in Z(E)$, then $\|T\| = \|T\|_r = \inf\{\lambda \ge 0 : |T| \le \lambda I\}$. By definition, $Z(E)$ is equal to the principal ideal generated by I in $\mathscr{L}_r(E)$. Actually, $Z(E)$ is a band in $\mathscr{L}_r(E)$ (see [9], section 142), and so we have the band decomposition

$$\mathscr{L}_r(E) = Z(E) \oplus Z(E)^d.$$

The corresponding band projection in $\mathscr{L}_r(E)$ onto $Z(E)$ will be denoted by $\mathscr{D}$. The projection onto the disjoint complement $Z(E)^d$ will be denoted by $\mathscr{D}_\perp$, i.e., $\mathscr{D}_\perp = I - \mathscr{D}$. In this paper we will call $\mathscr{D}(T)$ the *diagonal* of the operator T; $\mathscr{D}$ is called the *diagonal map.* The following theorem was proved for positive operators in [5, Proposition 2.3]. The proof differs only in some minor places of the proof given in [5], but as it is crucial for the approach presented here, we will present it in detail.

Theorem 2.1. *Let E be a Dedekind complete Banach lattice and let $S, T\in\mathscr{L}_r(E)$ such that $|S||T| = |T||S|$ and $r(|S|), r(|T|) \le 1$. Then there exists a (unique) sequence $\{F_n\}_{n=1}^{\infty}$ in $Z(E)$ such that*

$$\mathscr{D}(ST^{n-1}) = \sum_{k=0}^{n-1} F_{n-k}\mathscr{D}(T^k)$$

and $\sum_{k=1}^{n} |F_k| \le I$ *for all* $n \ge 1$.

Proof. Define the sequence $\{G_n\}_{n=1}^{\infty}$ in $\mathscr{L}_r(E)$ inductively by

$$G_1 = S, \qquad G_n = \mathscr{D}_\perp(G_{n-1})T \qquad (n \ge 2),$$

and let $F_n = \mathscr{D}(G_n)$. Note that $G_nT = G_{n+1} + F_nT$ for all $n \ge 1$. We first show that

$$\text{(1)} \qquad ST^{n-1} = G_n + \sum_{k=1}^{n-1} F_{n-k}T^k$$

for all $n \ge 2$. Since $ST = \mathscr{D}_\perp(S)T + \mathscr{D}(S)T = G_2 + F_1T$, (1) holds for $n = 1$. Assuming that (1) holds for some $n \ge 2$ we have

$$\begin{aligned} ST^n = ST^{n-1}T &= \left\{G_n + \sum_{k=1}^{n-1} F_{n-k}T^k\right\}T \\ &= G_nT + \sum_{k=1}^{n-1} F_{n-k}T^{k+1} \\ &= G_{n+1} + F_nT + \sum_{k=1}^{n-1} F_{n-k}T^{k+1} = G_{n+1} + \sum_{k=1}^{n} F_{n+1-k}T^k \end{aligned}$$

which is (1) for $n + 1$. From (1) it follows immediately that

$$\mathscr{D}(ST^{n-1}) = \sum_{k=1}^{n-1} F_{n-k}\mathscr{D}(T^k)$$

for all $n \ge 2$. It remains to prove that $\sum_{k=1}^{n} |F_k| \le I$ for all $n \ge 1$. To this end take $0 \le u \in E, \lambda > 1$ and put $w = R(\lambda, |S|)R(\lambda, |T|)u$. Since $|T|w \le \lambda w$, we have $|G_n|w \le \mathscr{D}_\perp(|G_{n-1}|)|T|w \le \lambda\mathscr{D}_\perp(|G_{n-1}|)w = \lambda(|G_{n-1}|w - |F_{n-1}|\,w)$ for all $n \ge 2$. We claim that

$$\text{(2)} \qquad |G_n|w \le \lambda^n w - \sum_{k=1}^{n-1} \lambda^k|F_{n-k}|w$$

for all $n \ge 2$. Indeed, it follows from $|S|w \le \lambda w$ that $|G_2|w \le \lambda(|G_1|w - |F_1|w) = \lambda(|S|w - |F_1|w) \le \lambda^2 w - \lambda|F_1|w$, which is (2) for $n = 2$. Now assume that (2) holds for some $n \ge 2$. Then

$$\begin{aligned} |G_{n+1}|w \le \lambda(|G_n|w - |F_n|w) &\le \lambda\left(\lambda^n w - \sum_{k=1}^{n-1}\lambda^k|F_{n-k}|w\right) - \lambda|F_n|w \\ &= \lambda^{n+1}w - \sum_{k=1}^{n} \lambda^k|F_{n+1-k}|w\,, \end{aligned}$$

and this proves the claim. It now follows from (2) that

$$|F_n| w \le |G_n| w \le \lambda^n w - \sum_{k=1}^{n-1} \lambda^k |F_{n-k}|\, w\,,$$

and hence

$$\sum_{k=1}^{n} \lambda^{n-k} |F_k|\, w \le \lambda^n w$$

for all $n \ge 2$, i.e.,

$$\left[\sum_{k=1}^{n} \lambda^{n-k} |F_k| - \lambda^n I\right]^+ = 0$$

for all $n \ge 2$. Since $u \le \lambda^2 w$, this implies that

$$\left[\sum_{k=1}^{n} \lambda^{n-k} |F_k| - \lambda^n I\right]^+ u = 0\,,$$

and hence $\sum_{k=1}^{n} \lambda^{n-k} |F_k| u \le \lambda^n u$ for all $n \ge 2$. This holds for any $\lambda > 1$, so we may conclude that $\sum_{k=1}^{n} |F_k| u \le u$ for all $0 \le u \in E$ and all $n \ge 2$, and this completes the proof of the theorem. □

Now we shall consider positive definite sequences in $Z(E)$. Let $(U_n)_{n\in\mathbb{Z}}$ be a positive definite sequence in $Z(E)$ with $U_0 = I$. Let $U(z) = \sum_{n\ge 0} U_n z^n$. Then by Theorem 1.6 we have that $\operatorname{Re} U(z) \ge \frac{1}{2} I$ for all $|z| < 1$. Hence $F(z) = 1 - \frac{1}{U(z)}$ satisfies $|F(z)| \le I$ for all $|z| < 1$ and $F(0) = 0$. Conversely, if $F(z)$ is an analytic function on $\{z : |z| < 1\}$ with values in $Z(E)$ and satisfies $|F(z)| \le I$ for all $|z| < 1$ and $F(0) = 0$, then $U(z) = \frac{1}{1-F(z)}$ satisfies $\operatorname{Re} U(z) \ge \frac{1}{2}$ for all $|z| < 1$. Writing $F(z) = \sum_{n=1}^{\infty} F_n z^n$ for $|z| < 1$ we can express the relation between $U(z)$ and $F(z)$ by means of the recurrence relation

$$(3) \qquad U_0 = 1\,, \qquad U_n = F_n + F_{n-1} F_1 + \dots + F_1 U_{n-1} \qquad (n \ge 1)\,.$$

From the above discussion it follows now that a sequence $(U_n)_{n\in\mathbb{Z}}$ in $Z(E)$ with $U_0 = I$ is positive definite if and only if the unique sequence F_n defined by the recurrence relation (3) defines an analytic function $F(z) = \sum_{n=1}^{\infty} F_n z^n$ for $|z| < 1$ such that $|F(z)| \le I$ for all $|z| < 1$. In particular it is a sufficient (but not necessary) condition to require that the sequence F_n satisfies $\sum_{n\ge 1} |F_n| \le I$ in order that the corresponding sequence U_n is positive definite. From this we obtain immediately by taking $S = T$ in Theorem 2.1 the following corollary.

Corollary 2.2. *Let E be a Dedekind complete Banach lattice and let $T \in \mathscr{L}_r(E)$ such that $r(|T|) = 1$. Then the sequence $\{\widehat{\mathscr{D}(T^n)}\}_{n\in\mathbb{Z}}$ is positive definite.*

Remark. It was observed by Toeplitz that positive definiteness of a sequence can be expressed in terms of non-negativity of a corresponding sequence of

Toeplitz determinants. In our situation this says that under the above hypotheses we have that

$$\begin{vmatrix} I & \mathscr{D}(T) & \mathscr{D}(T^2) & \dots & \mathscr{D}(T^n) \\ \overline{\mathscr{D}(T)} & I & \mathscr{D}(T) & \dots & \mathscr{D}(T^{n-1}) \\ \overline{\mathscr{D}(T^2)} & \overline{\mathscr{D}(T)} & I & \dots & \mathscr{D}(T^{n-2}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\mathscr{D}(T^n)} & \overline{\mathscr{D}(T^{n-1})} & \overline{\mathscr{D}(T^{n-2})} & \dots & I \end{vmatrix} \geq 0 .$$

for all $n \geq 1$.

[5] the above corollary was derived from Andô's inequality (see [2] for a proof of this inequality for matrices). As indicated in the introduction, here we shall now obtain Andô's inequality as an easy consequence of Corollary 1.7.

Corollary 2.3 (Andô's Inequality). *Let E be a Dedekind complete Banach lattice and let $T \in \mathscr{L}_r(E)$. Then $|TR(\lambda, T)| \leq |\lambda R(\lambda, T)|$ for all $|\lambda| > r(|T|)$.*

Proof. Without loss of generality we can assume that $r(|T|) = 1$. Since for $\lambda \in \varrho(T)$ we have $TR(\lambda, T) = \lambda R(\lambda, T) - I$, it is clear that $|\mathscr{D}_\perp[TR(\lambda, T)]| = |\mathscr{D}_\perp[\lambda R(\lambda, T)]|$. Hence the theorem is equivalent to the inequality $|\mathscr{D}[TR(\lambda, T)]| \leq |\mathscr{D}[\lambda R(\lambda, T)]|$ in $Z(E)$. From Corollary 1.7 it follows together with the above corollary that $|\sum_{n\geq 1}\mathscr{D}(T^n)z^n| \leq |\sum_{n\geq 0}\mathscr{D}(T^n)z^n|$ for all $|z| < 1$, which turns into the inequality $|\mathscr{D}[TR(\lambda, T)]| \leq |\mathscr{D}[\lambda R(\lambda, T)]|$ if we put $z = \frac{1}{\lambda}$. □

Let now $0 \leq T \in \mathscr{L}_r(E)$ be an arbitrary positive operator. Then we denote by $\mathscr{P}_T$ the band projection on the band $\{T\}^{dd}$ generated by T in $\mathscr{L}_r(E)$. Recall now that a positive operator T is called a Riesz homomorphism if $x \wedge y = 0$ in E implies $Tx \wedge Ty = 0$ and that T is called interval preserving if $T[0, x] = [0, Tx]$ for all $0 \leq x \in E$. Also recall that a positive operator T is called order continuous if $x_\tau \downarrow 0$ in order implies $Tx_\tau \downarrow 0$ in order.

Proposition 2.4. *Let E be a Dedekind complete Banach lattice and let $S \in \mathscr{L}_r(E)$. Then the following holds.*

(1) *If T is an order continuous Riesz homomorphism, then $\mathscr{P}_T(TS) = T\mathscr{D}(S)$.*
(2) *If T is an interval preserving operator, then $\mathscr{P}_T(ST) = \mathscr{D}(S)T$.*

Proof. We can assume that $S \geq 0$, the general case follows by linearity. Assume first that T is an order continuous Riesz homomorphism. Then left multiplication by T is a Riesz homomorphism on $\mathscr{L}_r(E)$ (see [1, Theorem 7.5]). Hence $\mathscr{P}_T(TS) = \sup_n (TS \wedge nT) = T \sup_n (S \wedge nI) = T\mathscr{D}(S)$. Hence (1) holds. The proof of (2) is completely similar, when one observes that in this case right multiplication by T is an order continuous Riesz homomorphism on $\mathscr{L}_r(E)$ (see [1, Theorem 7.4]). □

Corollary 2.5. *Let E be a Dedekind complete Banach lattice and let $S \in \mathscr{L}_r(E)$ with $r(|S|) = 1$. Then the following holds.*

(1) *If T is an order continuous Riesz homomorphism, then $\{\widehat{\mathscr{P}_T(TS^n)}\}_{n\in\mathbb{Z}}$ is positive definite.*

(2) *If T is an interval preserving operator, then $\{\widehat{\mathscr{P}_T(S^nT)}\}_{n\in\mathbb{Z}}$ is positive definite.*

Proof. From the above proposition it follows in case (1) that $\mathscr{P}_T(TS^n) = T\mathscr{D}(S^n)$ and in case (2) that $\mathscr{P}_T(S^nT) = \mathscr{D}(S^n)T$. The sequence $\{\widehat{\mathscr{D}(S^n)}\}_{n\in\mathbb{Z}}$ is now positive definite by Corollary 2.2. From Definition 1.1 it is immediate that the product from the right (or from the left) of a positive definite sequence in $\mathscr{L}_r(E)$ with a positive operator is again positive definite in $\mathscr{L}_r(E)$. Hence the corollary follows. □

3. Diagonal elements of contractions on ℓ_p

In this section we indicate another way of obtaining positive definite scalar sequences from the "diagonal elements" of powers of an operator. Let $1 \leq p \leq \infty$ and let I denote an arbitrary non-empty set. Denote by e_i the sequence in $\ell_p(I)$ with i^{th}-coordinate equal to 1 and all other coordinates equal to 0. By e_i^*we denote the same sequence, considered as element of the dual space of $\ell_p(I)$. If now T is a bounded operator on $\ell_p(I)$, then one can consider $\langle T_{e_i}, e_i^*\rangle$ as a diagonal element of T.

Theorem 3.1. *Let T be a linear contraction on $\ell_p(I)$. Then for each $i \in I$ the sequence $\{\widehat{\langle T^n e_i, e_i^*\rangle}\}_{n\in\mathbb{Z}}$ is positive definite.*

Proof. Let $|z| < 1$ and define $x = \sum_{n\geq 0} T^n e_i z^n$. Then $x - e_i = zTx$. Hence $\|x - e_i\|_p \leq \|x\|_p$. Assume first $1 \leq p < \infty$. Then it follows from $\|x - e_i\|_p \leq$ $\leq \|x\|_p$ that $|x(i) - 1| \leq |x(i)|$ and thus $\operatorname{Re}\langle x,e_i^*\rangle \geq \frac{1}{2}$. It follows now from the scalar version of Theorem 1.6 that $\{\widehat{\langle T^n e_i, e_i^*\rangle}\}_{n\in\mathbb{Z}}$ is positive definite. In case $p = \infty$ one can consider T^* on ℓ_∞ and by using that $e_i^*$is an atom one can proceed as above in this case. □

In case H is Hilbert space and $x \in H$, then by representing H as a space $\ell_2(I)$ with $x = e_i$ for some i, we obtain the known result that $\{\widehat{\langle T^n x, x\rangle}\}_{n\in\mathbb{Z}}$ is positive definite for any contraction T on H (see e.g. [4, Proposition 2.3.1]). Based on the Hilbert space case, one could conjecture that in case E is a Banach space and T is a contraction on E, then for $x \in E$ with $\|x\| = 1$ and $f \in E^*$ with $\|f\| = \langle f, x\rangle = 1$ the sequence $\{\widehat{\langle T^n x, f\rangle}\}$ is positive definite. This is not the case, as the following example shows.

Example. Let $E = \mathbb{R}^2$ with

$$\|(x,y)\| = \begin{cases} \max\{|x|,|y|\}, & \text{in case } xy \geq 0 \\ |x| + |y|, & \text{otherwise.} \end{cases}$$

Define T on E by $T(x,y) = (x - y, x - y)$. It is easy to see that $\|T\| = 1$ with respect to this norm on E. Consider now the complexification $E_{\mathbb{C}}$ of E, where

$$\|x + iy\|_{\mathbb{C}} = \sup_{\theta} \|(\cos\theta)x + (\sin\theta)y\| .$$

One can verify easily that the complexification of T on $E_{\mathbb{C}}$ has the same norm as T on E and also that if $e_1 = (1,0)$ and $e_1^* = (1,0)$, then $\|e_1\|_{\mathbb{C}} = 1$ and $\|e_1^*\|_{\mathbb{C}} = 1$, but the sequence $\{\widehat{\langle T^n e_1, e_1^* \rangle}\}$ is not positive definite, e.g. by Proposition 1.2, since $1 + e^{i\theta} + e^{-i\theta} = 1 + 2\cos\theta$ is not nonnegative for all θ.

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