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Stable Smooth and Extreme Points, and Reflexivity

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The property "for any equivalent norm each extreme point of the unit ball is an extreme point of the second dual unit ball" characterizes reflexive Banach spaces. If we consider smooth points instead of extreme points, the corresponding property characterizes Grothendieck spaces. Some other characterizations of reflexivity in terms of renormings are given for WCG Banach spaces.

Introduction

Let $\mathfrak{P}$ be a geometric property of the unit sphere in Banach spaces. We shall say that a point $x \in S_X$ is a *stable point with* $\mathfrak{P}$ if x has $\mathfrak{P}$ and its canonical image $\hat{x} \in S_{X^{**}}$ also has $\mathfrak{P}$. (So we shall speak about "stable smooth points", "stable extreme points", etc..)

Of course, any property is stable in reflexive spaces. A natural question is: for which properties $\mathfrak{P}$ the condition.

(*) *for any equivalent norm each point that has* $\mathfrak{P}$ *is a stable point with* $\mathfrak{P}$ characterizes reflexivity? It is known that this is not the case if $\mathfrak{P}$ = "to be a Fréchet smooth point" or $\mathfrak{P}$ = "to be a LUR point", since in every Banach space each Fréchet smooth point/LUR point is Fréchet smooth/LUR in the second dual (see Appendix).

In the present paper we prove that (*) characterizes reflexivity if $\mathfrak{P}$ = "to be an extreme point of the unit ball", but (*) does not characterize reflexive spaces if $\mathfrak{P}$ = "to be a smooth point" (it characterizes Grothendieck spaces, i.e. such spaces that each *weak**-null sequence in the dual is *weak*-null). The following theorem is the main result of this paper.

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Theorem 0.1.

For a Banach space X the following assertions are equivalent.

(i) X is reflexive.

(ii) Each extreme point of the unit sphere of any equivalent norm on X, is a stable extreme point.

(iii) B_{X^*} is sequentially weak compact and each smooth point of the unit sphere of any equivalent norm on X, is a stable smooth point.*

(iv) X is WCG (= weakly compactly generated) and each point of the unit sphere of any equivalent rotund norm on X is a stable extreme point.

(v) X is WCG and each point of the unit sphere of any equivalent smooth norm on X, is a stable smooth point.

(vi) X is WCG and each point of the unit sphere of any equivalent dual rotund norm on X^*, is a stable extreme point.

(vii) X is WCG and each equivalent rotund norm on X has a smooth dual.

Moreover, the following condition (iii′) holds if and only if X is a Grothendieck space.

(iii′) Each smooth point of the unit sphere of any equivalent norm on X, is a stable smooth point.

We were inspired by a paper by B. V. Godun [3] whose proofs implicitely contain the equivalence of (i), (iii), (v), (vi) for separable Banach spaces.

In the first section. we generalize Godun's method to obtain Theorem 1.2 and Theorem 1.3 that give sufficient conditions to construct equivalent rotund norms with a non-stable extreme point. These conditions are satisfied for nonreflexive WCG spaces (Proposition 1.4) and hence (i), (iv), (v), (vi), (vii) are equivalent. The author does not know whether the conditions of Theorem 1.2 and Theorem 1.3 are satisfied in more general spaces (Problem 1.5); on the Winter School in Abstract Analysis in Poděbrady 1993 he announced such a result but later he found that his proof was not correct.

The main result of the second section is Theorem 2.2 which implies the equivalence of (i) and (ii). We use the results of the first section and an extension lemma for norms. An open problem on rotund extensions of rotund norms from a subspace to the whole space (Problem 2.3), arises.

The third section contains the proof that (iii′) characterizes Grothendieck spaces (Theorem 3.3). This implies the equivalence of (i) and (iii) since any Grothendieck space with a sequentially *weak** compact dual ball is reflexive by the Eberlein-Šmulian theorem.

We add short proofs of the stability of Fréchet smooth points and LUR points in Appendix.

Let us state some preliminaries. All Banach spaces in the present paper are real. by B_X and S_X we denote the closed unit ball and the unit sphere of the Banach space X. The duality mapping $J\colon X \to 2^{X^*}$ is defined by $J(x) = \{x^* \mid \langle x, x^*\rangle = \|x\|\,\|x^*\| \text{ and } \|x\| = \|x^*\|\}$. It is well known that the norm $\|\,.\,\|$ is Gâteaux differentiable at a point $x \in X \setminus \{0\}$ if and only if $J(x)$ is a singleton (cf. [1]). If this is true we call x a *smooth point* of X. We shall also say that X or $\|.\|$ is smooth at x. The Banach spce X is called *smooth* if it is smooth at each point of its unit sphere. the space X is *rotund* if its unit sphere S_X does not contain any nontrivial line segment, or equivalently, if each point of S_X is an extreme point of B_X. We denote dual norms by the same symbol as the norms. By $\hat{x}$, $\hat{A}$, $\hat{x}^*$ we denote the canonical image of $x \in X$ in the second dual X^{**}, of a set $A \subset X$ in X^{**}, of $x^* \in X^*$ in X^{***}, respectively. The closed and open line segment with endpoints x, y are denoted by (x, y) and $[x, y]$; thus $(x, y) = \{\lambda x + (1 - \lambda)y \mid 0 < \lambda < 1\}$ and $[x, y] = \{\lambda x + (1 - \lambda)y \mid 0 \leqq \lambda \leqq 1\}$. The symbol $B(x, r)$ denotes the closed ball of radius r, centered at x. By the *weak** continuity we mean the *weak**-to-*weak** continuity.

1. Godun's method

In [3] B. V. Godun proved that each nonreflexive separable Banach space can be renormed equivalenly to have rotund dual but not all points of the dual unit sphere are stable extreme points. A generalization of the methods from [3] is presented in this section. It is based on the following proposition.

Proposition 1.1. *Let T be a bounded linear operator from a Banach space $(X, \|\cdot\|)$ into a Banach space $(Y, |\cdot|)$. Suppose $x_0 \in X$, $T_{x_0} \neq 0$, $\|x_0\| < |T_{x_0}| = 1$. Then*

$$\pi(x) := \max\{\|x\|, |Tx|\}$$

is an equivalent norm on X that has the following properties.

(a) *x_0 is an extreme point of $B_{(X,\pi)}$ if and only if T is one-to-one and $\frac{Tx_0}{|Tx_0|}$ is an extreme point of $B_{(T(X), |\cdot|)}$.*

(b) *If both $(X, \|\cdot\|)$ and $(T(X), |\cdot|)$ are rotund and T is one-to-one, then (X, π) is rotund.*

(c) *If both $(X, \|\cdot\|)$ and $(Y, |\cdot|)$ are dual and T is weak* continuous, then (X, π) is dual.*

(d) *The second dual norm of π is given by $\pi(x^{**}) = \max\{\|x^{**}\|, |T^{**}x^{**}|\}$.*

Proof. It is easy to see that $\|x\| \leqq \pi(x) \leqq m\|x\|$ for all $x \in X$, where $m = \max\{1, \|T\|\}$.

(a) First note that the assumptons imply $\pi(x_0) = |T_{x_0}| = 1$.
Suppose T is one-to-one and $\frac{Tx_0}{|Tx_0|}$ is extreme in $B_{(T(X), |\cdot|)}$. If $y \in X$ is such that $\pi(x_0 \pm y) = \pi(x_0)$, take $\varrho > 0$ so small that $|T(x_0 \pm \varrho y)| > \|x_0 \pm \varrho y\|$ (this is possible by continuity). Then $|Tx_0 \pm \varrho Ty| = |Tx_0|$ by the definition of π. The extremality assumption implies $Ty = 0$; consequently $y = 0$ since T is one-to-one.

Now, suppose that T is not one-to-one or $\frac{Tx_0}{|tx_0|}$ is not an extreme point of $B_{(T(X),|\cdot|)}$. In both situations there exists $y \neq 0$ such that $|Tx_0 \pm Ty| = |Tx_0|$. Similarly as above, $\pi(x_0 \pm \varrho y) = \pi(x_0)$ for some $\varrho > 0$. This implies that x_0 is not an extreme point of $B_{(X,\pi)}$.

(b) Observe that $B_{(X,\pi)}$ is the intersection of two rotund conves sets $B_{(X,\|\cdot\|)}$ and $T^{-1}(B_{(T(X),|\cdot|)})$.

(c) $B_{X,\|\cdot\|)}$ and $B_{(Y,|\cdot|)}$ are *weak** closed, hence also $B_{(X,\pi)} = B_{(X,\|\cdot\|)} \cap \cap T^{-1}(B_{(Y,|\cdot|)})$ is, if T is *weak** continuous.

(d) Consider $F = \{(x, Tx) \mid x \in X\}$ as a subspace of $E = (X, \|\cdot\|) \oplus_\infty (Y, |\cdot|)$. Then (X, π) is isometric to F, and

$$F^{**} \equiv (F^\perp)^\perp = \{(x^{**}, T^{**}x^{**}) \mid x^{**} \in X^{**}\} \subset E^{**} \equiv (X^{**}, \|\cdot\|) \quad \oplus_\infty (Y^{**}, |\cdot|).$$

This implies the formula for the second dual norm of (X, π).

As a consequence we obtain two theorems on the existence of equivalent rotund or smooth norms with a non-stable extreme or smooth point.

Theorem 1.2. *Let X be a Banach space that admits a rotund norm. Suppose that there exist a Banach space Y and a one-to-one bounded linear operator $T: X \to Y$ such that $T(X)$ admits an equivalent rotund norm and T^{**} is not one-to-one. Then there exists an equivalent norm π on X such that*

(i) (X, π) *is rotund,*
(ii) $B_{(X,\pi)}$ *contains a non-stable extreme point,*
(iii) (X^*, π) *is not smooth.*

Proof. Let $\|\cdot\|$ and $|\cdot|$ be rotund norms on X and $T(X)$, respectively. It is a well-known fact that $|\cdot|$ can be extended to an equivalent norm on Y (for example, it is possible to extend the norm to $\overline{T(X)}$ by continuity and then proceed as in Lemma 2.1). Take any nonzero $x_0 \in X$. It is possible to assume that $\|x_0\| < |Tx_0| = 1$ (consider suitable multiples of the norms). Define π as in Proposition 1.1. Then π is rotund by *(b)*. By *(d)* and *(a)*, $\hat{x}_0$ is not an extreme point in $B_{X,\pi)}$. Let u^{**}, $v^{**} \in B_{(X^{**},\pi)}$ be such that $\hat{x}_0 \in (u^{**}, v^{**})$. Consider any $x^* \in J(x_0)$ where J is the duality mapping of (X, π). Then $\hat{x}_0 \in J(x^*)$ implies $[u^{**}, v^{**}] \subset J(x^*)$; consequently (X^*, π) is not smooth at x^*.

Theorem 1.3. *Let X be a Banach space that admits an equivalent norm whose dual is rotund. Suppose that there exist a Banach space Y with Y^* rotund and a one-to-one weak* continuous linear operator $T: X^* \to Y^*$ such that T^{**} is not one-to-one. Then there exists an equivalent norm π on X such that*

(i) (X, π) *is smooth,*
(ii) (X^*, π) *is rotund,*
(iii) $B_{(X,\pi)}$ *contains a non-stable smooth point,*
(iv) $B_{(X^*,\pi)}$ *contains a non-stable extreme point.*

Proof. Let $\|\cdot\|$ and $|\cdot|$ be equivalent norms on X and Y (respectively) such that the duals are rotund. Taking suitable multiples of the norms, it is possible to suppose that $A := \{x^* \in X^* \mid \|x^*\| < |Tx^*|\} \neq \emptyset$. The formula

$$\pi(x^*) := \max\{\|x^*\|, |Tx^*|\}$$

defines an equivalent dual rotund norm on X^* by Proposition 1.1. Hence its predual norm on X is smooth (cf. [1]). By the Bishop-Phelps theorem (cf. [6]), there exists $x_0^* \in A \cap S_{(X^*,\pi)}$ which attains its p-norm at some $x_0 \in S_{(X,\pi)}$. By Proposition 1.1 *(d), (a),* the point x_0^* is not an extreme point of $B_{(X^{***},\pi)}$. As in the proof of Theorem 1.2, $B_{X^{**},\pi)}$ is not smooth at the point $\hat{x}_0 \in J(x_0^*)$.

Proposition 1.4. *Any nonreflexive WCG Banach space X satisfies the assumptions of the theorems 1.2 and 1.3.*

Proof. 1. By the Eberlein-Šmulian theorem, X contains a nonreflexive separable subspace. By a property of WCG spaces (cf. [1], p. 149, Theorem 3; or [5], Theorem 2.1) the subspace is contained in another separable subspace Z which is complemented in X. Clearly, Z is nonreflexive. Choose an arbitrary $z_0^{**} \in Z^{**} \setminus \hat{Z}$. It is easy to see that there exists a sequence $\{z_n^*\} \subset Z^*$ which is total on Z and such that $\langle z_n^*, z_0^{**}\rangle = 0$ and $\|z_n^*\| = 2^{-n}$ for all n (use the w^*-density of $\ker(z_0^{**})$ in Z^* and the existence of a dense countable subset of Z). Let $Q \subset X$ be a closed linear subspace with $X = Z \oplus Q$. Consider Q equipped with an equivalent rotund norm; such a norm exists since Q is a subspace of a WCG space (cf. [1]). Put $Y = \ell_2 \oplus_2 Q$ and define $T: X \to Y$ by $T(z, q) = (\{\langle z, z_n^*\rangle\}_{n=1}^{\infty}, q)$. Then Y is rotund and T is a one-to-one bounded linear operator. The second adjoint

$$T^{**}: X^{**} \equiv Z^{**} \oplus Q^{**} \to Y^{**} \equiv \ell_1 \oplus Q^{**}$$

is given by $T^{**}(z^{**}, q^{**}) = (\{\langle z_n^*, z^{**}\rangle\}_{n=1}^{\infty}, q^{**})$. It is not one-to-one since $T^{**}(z_0^{**}, 0) = (0, 0)$.

2. If X is WCG, there exist a reflexive Banach space Y and a bounded linear operator S: $Y \to X$ such that $\overline{S(Y)} = X$. Then the operator $T := S^*: X^* \to Y^*$ is one-to-one and *weak** continuous. Moreover, its second adjoint $T^{**}: X^{***} \to Y^{***} \equiv Y^*$ is zero on $\hat{X}^{\perp}$. Indeed, for any $x^{***} \in \hat{X}^{\perp}$ and $y \in Y$ we have

$$\langle T^{**}x^{***}, y\rangle = \langle x^{***}, T^*y\rangle = \langle x^{***}, S^{**}y\rangle = \langle x^{***}, \widehat{Sy}\rangle = 0 .$$

Thus T is not one-to-one if X is not reflexive. To complete the proof it is sufficient to observe that X and Y admit equivalent norms whose duals are rotund (cf. [1]).

It follows from Proposition 1.4 that the conditions (i), (iv), (v), (vi) from Theorem 0.1 are equivalent. Indeed, if X is a nonreflexive WCG Banach space, Theorem 1.2 (Theorem 1.3, respectively) implies that X does not satisfy any of the conditions (iv) and (vii) ((v) and (vi), resp.) from Theorem 0.1 It is obvious that if X is reflexive, all conditions from Theorem 0.1 are satisfied.

Problem 1.5.

(α) Does any nonreflexive rotund Banach space X satisfy the assumptions of Theorem 1.2?

(β) Does any nonreflexive Banach space X with X^* rotund satisfy the assumptions of Theorem 1.3?

An affirmative answer to (α) or (β) would imply that the condition "X is WCG" in Theorem 0.1 (iv), (vii) or Theorem 0.1 (v), (vi) (respectively) can be replaced by a condition on the existence of an equivalent rotund norm on X or dual rotund norm on X^* (respectively).

2. Stable extreme points

The aim of this section is to prove that the condition that for any equivalent renorming all extreme points of the unit ball are stable extreme points, is equivalent to the reflexivity of the space in question. Main tools are results of the first section and the following extension lemma. The construction of the required equivalent norm in the proof of Lemma 2.1 is geometrically obvious.

Lemma 2.1. *Let Z be a closed linear subspace of a Banach space $(X, \|\cdot\|)$. Let $|\cdot|$ be an equivalent rotund norm on Z. Then $|\cdot|$ can be extended to an equivalent norm on X such that each point of $S_{(Z,|\cdot|)}$ is an extreme point of $B_{(X,|\cdot|)}$.*

Proof. Denote $B = B_{(X,\|\cdot\|)}$, $B_Z = B_{(Z,|\cdot|)}$. Without any loss of generality it is possible to suppose that $2B \cap Z \subset B_Z$. Put

$$K = \overline{\mathrm{co}}\,(B \cup B_Z)\,, \qquad \tilde{K} = \overline{\mathrm{co}}\,(2B \cup B_Z)\,.$$

It is elementary to see that K, $\tilde{K}$ are symmetric closed convex bodies with

$$K \cap Z = \tilde{K} \cap Z = B_Z\,.$$

Thus K is the unit ball of an equivalent norm on X that extends $|\cdot|$.

It remains to show that any point $z \in S_{(Z,|\cdot|)}$ is an extreme point of K. Suppose not, i.e. there exist $x, y \in K$ such that $z \in (x, y)$. Observe that x, y cannot both belong to Z since B_Z is rotund. Then necessarily x, y both belong to $K \setminus Z$. The point z is a boundary point of $\tilde{K}$, hence there exist $u^* \in S_{(X,\|\cdot\|)}$ that supports $\tilde{K}$ at z. Then u^* supports K at z, and hence at x and at y, too. Therefore

$$\begin{aligned} s := \langle x, u^*\rangle = \langle y, u^*\rangle = \langle z, u^*\rangle &= \sup\langle K, u^*\rangle = \sup\langle \tilde{K}, u^*\rangle \\ &\geqq \sup\langle 2B, u^*\rangle = 2 \end{aligned}$$

By the Hahn-Banach theorem there exists $v^* \in Z^\perp$ such that $\|v^*\| = 1$ and $\langle x, v^*\rangle > 0$. By the definition of K there exist nonnegative numbers λ_n, μ_n and points $z_n \in B_Z$, $b_n \in B$ such that

$$\lambda_n + \mu_n = 1 \text{ for all } n, \text{ and } \lambda_n z_n + \mu_n b_n \to x\,.$$

Then $\nu := \langle x, v^* \rangle = \lim \langle \lambda_n z_n + \mu_n b_n, v^* \rangle = \lim \mu_n \langle b_n, v^* \rangle \leqq \liminf \mu_n$. But this, together with $1 - s \leqq -1$, implies

$$\begin{aligned} s = \langle x, u^* \rangle &= \lim (\lambda_n \langle z_n, u^* \rangle + \mu_n \langle b_n, u^* \rangle) \\ &\leqq \liminf (\lambda_n s + \mu_n) = \liminf (s + (1 - s)\mu_n) \\ &\leqq \liminf (s - \mu_n) \leqq s - \liminf \mu_n \\ &\leqq s - \nu < s, \end{aligned}$$

a contradiction.

Theorem 2.2. *Let X be a nonreflexive Banach space, $x_0 \in X \setminus \{0\}$. Then there exists an equivalent norm π on X such that x_0 is a non-stable extreme point of $B_{(X,\pi)}$.*

Proof. Choose a nonreflexive separable subspace Z of X such that $x_0 \in Z$. By Propositon 1.4 and Theorem 1.2 (and its proof), there exists an equivalent rotund norm π on Z such that x_0 is a non-stable extreme point of $B_{(Z,\pi)}$. By Lemma 2.1, there exists an extension of π to an equivalent norm on X such that x_0 is an extreme point of $B_{(X,\pi)}$. Clearly, $\hat{x}_0$ is not an extreme point of $B_{(X^{**},\pi)}$ since it is not an extreme point of $B_{(Z^{**},\pi)}$.

As a direct consequence of Theorem 2.2 we obtain the equivalence of (i) and (ii) in Theorem 0.1.

Problem 2.3. Let Z be a (separable) subspace of a rotund Banach space X. Let $|\cdot|$ be an equivalent rotund norm on Z. Does there exist an equivalent rotund norm on X that extends $|\cdot|$? (If so, it would be possible to replace the words "X is WCG" in Theorem 0.1 (iv), (vii) by "X has an equivalent rotund norm"; with the same proof as that of Theorem 2.2.)

3. Stable smooth points

The situation of smooth points is different from that of extreme points. We shall prove that the condition that for any equivalent renorming all smooth points are stable smooth points, is satisfied if and only if the space is a Grothendieck space.

Definition 3.1. A Banach space X is called a *Grothendieck space* if the *weak** and *weak* convergence of sequences in X^* are the same.

Each reflexive space is obviously a Grothendieck space. For example ℓ_∞ is a Grothendieck space (cf. [2], p. 103, Theorem 15). It is easy to see that a Grothendieck space is reflexive if and only if its dual unit ball is sequentially *weak** compact. (This condition is satisfied e.g. for Banach spaces that admit an equivalent smooth norm, [4].)

Remark 3.2. It is elementary to see that the direct sum of two Grothendieck spaces is again a Grothendieck space. Consequently, a Banach space is a Grothendieck space if the *weak** and *weak* convergence coincide for sequences contained in a given *weak** closed hyperplane in the dual.

Theorem 3.3. *A Banach space X is a Grothendieck space if and only if for each equivalent norm π on X each smooth point of (X, π) is stable.*

Proof. *a)* Let $x \in S_X$ be a smooth point which is not stable. There exist $x^* \in J(x)$ and $x^{***} \in J(\hat{x})$ such that $x^{***} \neq \hat{x}^*$. Choose $y^{**} \in X^{**}$ and $p \in \mathbf{R}$ such that $\langle y^{**}, x^{***}\rangle < p < \langle x^*, y^{**}\rangle$. The *weak** density of B_{X^*} in $B_{X^{***}}$ implies that for any positive integer n there exist $z_n^* \in B_{X^*}$ such that $\langle z_n^*, y^{**}\rangle < p$ and $\langle x, z_n^*\rangle > \frac{n-1}{n}$. Then for any *weak** limit point z^* of $\{z_n^*\}$ we have $\langle x, z^*\rangle = 1$, in other words $z^* \in J(x) = \{x^*\}$. Consequently $z_n^* \xrightarrow{w^*} x^*$. But $\{z_n^*\}$ cannot converge *weakly* to x^* since $\langle x^* - z_n^*, y^{**}\rangle > \langle x, y^{**}\rangle - p$ for all n.
b) Choose arbitrary $x \in S_X$, $x^* \in J(x)$. Suppose X is not a Grothendieck space. By Remark 3.2, there exists a sequence $\{y_n^*\} \in X^*$ such that $\langle x, y_n^*\rangle = 0$ for all n, $y_n^* \xrightarrow{w^*} 0$ and $\{y_n^*\}$ is not *weak*-null. Choose numbers $0 < a_0 < a_1 < a_2 < \ldots < 1$ so that $a_n \to 1$. Put $M = \sup_n \|y_n^*\|$ and $x_n^* = a_n(x^* + y_n^*)(n \geq 1)$. Then $\{x_n^*\}$ converges *weak** but not weakly to x^*, and $\langle x, x_n^*\rangle = a_n$.
Define

$$K = \overline{\text{co}}^{w^*}[a_0 B_{X^*} \cup \{\pm x^*\} \cup \{\pm x_n^*\}_{n \geq 1}].$$

Then K is the unit ball of an equivalent dual norm π on X^*, and $a_0 B_{X^*} \subset K \subset (1 + M) B_{X^*}$. Moreover $\pi(x) = \pi(x^*) = 1$ since $x^* \in K$, $\sup\langle x, K\rangle \leq 1$ and $\langle x, x^*\rangle = 1$.

Claim: π is smooth at x. To prove this, it suffices to show the implication

$$[z^* \in K, \langle x, z^*\rangle = 1] \Rightarrow z^* = x^*.$$

Suppose, on the contrary, that there is $z^*\ K \setminus \{x^*\}$ with $\langle x, z^*\rangle = 1$. There exist a vector $v \in S_X$ and real numbers p, q such that $\langle v, z^*\rangle < p < q < \langle v, x^*\rangle$. Since x^* is the *weak** limit of $\{x_n^*\}$, there exists an index n_0 such that $q < \langle v, x_n^*\rangle$ whenever $n > n_0$. For simplicity denote

$$C = \text{co}[a_0 B_{X^*} \cup \{-x^*\} \cup \{-x_k^*\}_{k \geq 1} \cup \{x_n^*\}_{1 \leq n \leq n_0}],$$

$$D = \text{co}[\{x^*\} \cup \{x_n^*\}_{n > n_0}].$$

Then we have

$$\sup\langle x, C\rangle = a_{n_0}, \qquad \inf\langle v, D)\rangle \geq q,$$

$$C - D \subset 2K \subset 2(1 + M) B_{X^*}, \qquad \overline{\text{co}(C \cup D)} = K.$$

Hence for any positive integer n there exists $z_n^* \in \mathrm{co}\,(C \cup D)$ such that $\langle v, z_n^* \rangle < p$ and $\langle x, z_n^* \rangle > a_n$. It is possible to write $z_n^* = \lambda_n c_n^* + (1 - \lambda_n)\, d_n^*$ where $0 \leqq \lambda_n \leqq 1$, $c_n^* \in C$, $d_n^* \in D$. Let us compute

$$a_n < \langle x, z_n^* \rangle \leqq \lambda_n \sup \langle x, C \rangle + (1 - \lambda_n) \sup \langle s, K \rangle =$$
$$= \lambda_n a_{n_0} + (1 - \lambda_n) = \lambda_n (a_{n_0} - 1) + 1\,.$$

Consequently $0 \leqq \lambda_n < \frac{1 - a_n}{1 - a_{n_0}}$, which implies $\lambda_n \to 0$. Take m so big that $q - 2(1 + M)\lambda_m > p$. Then we have

$$\langle v, z_m^* \rangle = \lambda_m \langle v, c_m^* \rangle + (1 - \lambda_m)\langle v, d_m^* \rangle \geqq \langle v, d_m^* \rangle - \lambda_m |\langle v, c_m^* - d_m^* \rangle|$$
$$\geqq q - 2(1 + M)\lambda_m > p > \langle v, z_m^* \rangle\,,$$

a contradiction. We have proved the Claim.
We know that $\{x_n^*\}$ does not converge to x^* in the *weak* topology on X^*, hence $\{\hat{x}_n^*\}$ does not converge to $\hat{x}^*$ in the *weak** topology on X^{***}. Thus there exists a *weak** limit point x^{***} of the sequence $\{\hat{x}_n^*\}$, diferent from $\hat{x}^*$. Then x^{***} belongs to the *weak** closure of $\hat{K} \subset X^{***}$, i.e. to the third dual unit ball of our new equivalent norm. Consequently $\{x^{***}, \hat{x}^*\} \subset J(\hat{x})$ since $\langle \hat{x}, x^{***} \rangle = 1$. This implies that $\hat{x}$ is not a smooth point in the second dual.

Theorem 3.3 completes the proof of Theorem 0.1.

4. Appendix

We present a simple proof that some properties are always stable. These results are not original. The stability of Fréchet-smooth points is implicitely contained in Šmulian's work [8], p. 549, A and B. The stability of LUR points was proved in [9] in a different way.

Definition 4.1. Let X be a Banach space, $x \in S_X$, $x^* \in S_{X^*}$.
(a) x is called a *LUR point* if $y_n \to x$ whenever $\|y_n\| \leqq 1$ and $\|\frac{x + y_n}{2}\| \to 1$.
(b) x is *strongly exposed by* x^* if $x^* \in J(x)$ and $\operatorname{diam}\{y \in B_X \mid \langle y, x^* \rangle > 1 - \delta\} \to 0$ as $\delta \to 0+$. (If this is true, we say that x is a *strongly exposed point* of B_X and that B_X is *strongly exposed by* x^*.)
(c) x is a *Fréchet-smooth point* if the norm of X is Fréchet differentiable at x.

Observation 4.2.
(α) The definition of a LUR point can be written equivalently in the following ways:

$$\forall \varepsilon > 0\, \exists \delta > 0 [y \in B_X, \|x + y\| > 2(1 - \delta) \Rightarrow \|x - y\| \leqq \varepsilon]\,,$$
$$\forall \varepsilon > 0\, \exists \delta > 0 B_X \cap [X \setminus B(-x, 2 - 2\delta)] \subset B(x, \varepsilon)\,,$$
$$\forall \varepsilon > 0\, \exists \delta > 0 B_X \subset B(-x, 2 - 2\delta) \cup B(x, \varepsilon)\,.$$

(β) For any two subsets A, B of a topological space we have $\overline{A \cup B} = \bar{A} \cup \bar{B}$, and $A \cap \bar{B} \subset \overline{A \cap B}$ if A is open.

Lemma 4.3. *Let X be a Banach space, $x, z \in X$, $x^* \in X^*$, $\delta > 0$, $\varepsilon > 0$. Then*
(i) $B_X \subset B(z, \delta) \cup B(x, \varepsilon)$ *implies* $B_{X^{**}} \subset B(\hat{z}, \delta) \cup B(\hat{x}, \varepsilon)$,
(ii) $\operatorname{diam}\{y \in B_X \mid \langle y, x^* \rangle > 1 - \delta\} = \operatorname{diam}\{y^{**} \in B_{X^{**}} \mid \langle x^*, y^{**} \rangle > 1 - \delta\}$.

Proof. *(i)* $B_{X^{**}} = \overline{B_X}^{w^*} \subset \overline{B(z, \delta) \cup B(x, \varepsilon)}^{w^*} = B(\hat{z}, \delta) \cup B(\hat{x}, \varepsilon)$ by Observation 4.2(β).
(ii) Denote $A := \{y \in X \mid \langle y, x^* \rangle > 1 - \delta\}$, $A_* := \{y^{**} \in X^{**} \mid \langle x^*, y^{**} \rangle > 1 - \delta\}$. The inequality $\operatorname{diam}(A \cap B_X) \leqq \operatorname{diam}(A_* \cap B_{X^{**}})$ is obvious. To show the opposite inequality, it is sufficient to note that

$$\operatorname{diam}(A_* \cap B_{X^{**}}) = \operatorname{diam}(A_* \cap \overline{B_X}^{w^*}) \leqq \operatorname{diam}(\overline{A_* \cap B_X}^{w^*}) = \operatorname{diam}(A \cap B_X)$$

by Observation 4.2 (β).

Corollary 4.4. *The LUR points, the strongly exposed points, and the Fréchet-smooth points are always stable.*

Proof. The stability of the LUR points and the strongly extreme points follows directly from Lemma 4.3 and Observation 4.2 (α). By the well-known Šmulian's test of Fréchet smoothness [7], x is a Fréchet smooth point if and only if B_{X^*} is strongly exposed by $\hat{x}$. Then, by Lemma 4.3 (ii), $B_{X^{***}}$ is strongly exposed by $\hat{\hat{x}}$. Consequently, $\hat{x}$ is a Fréchet-smooth point by Šmulian's test.

The reader will be able to prove in the same simple way that also other properties that can be defined by balls or slices (e.g. denting points), are stable in any Banach space.

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