Charles Stegall

Concerning a certain σ-algebra in compact Hausdorff spaces

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 34 (1993), No. 2, 113--116

Persistent URL: http://dml.cz/dmlcz/702001

Concerning a Certain σ-Algebra in Compact Hausdorff Spaces

C. STEGALL

Linz*)

Received 14 April 1993

Let B be a Baire topological space and Φ a minimal upper semicontinuous compact valued map defined on B with values in T. Define $\mathbf{F}(\Phi)$ to be the collection of subsets of T defined by $E \in \mathbf{F}(\Phi)$ if and only if
(i) $\{b \in B: \Phi(b) \cap E \neq \emptyset\}$ is a Baire Property subset of B and
(ii) $\{b \in B: \Phi(b) \cap E \neq \emptyset$ and $\Phi(b) \cap (T \setminus E) \neq \emptyset\}$ is a set of the first category.
In **[S1]** the following is proved.

Theorem 0. *The collection* $\mathbf{F}(\Phi)$ *is a* σ*-algebra that contains the Borel subsets of* T *and is stable under the Souslin operation.*
Here we prove that:

Theorem I. *If* T *is a compact space,* $F \subseteqq T$ and $F \in \mathbf{F}(\Phi)$ *for any* Φ *then* F *is a Baire property subset of* T.
Necessary to the proof of the above, other than a few easy permutations of old and easy results, is the following result (also in **[S1]**). Actually, as pointed out in **[M]**, the relevant property of the mapping g is that if $N \subseteqq R$ is nowhere dense then $g^{-1}(N)$ is also nowhere dense; the proof in **[S1]** does this also. In our applications here, we make take $C = T$ and p the identity. If, in addition, all spaces are completely regular, we know that H is Čech-complete and is a G_δ subset of R.

Theorem II. *Suppose that* C *is Čech complete and* $p: C \to T$ *is perfect onto and* T *is a dense subspace of* S. *Suppose* $g: S \to R$ *is continuous, open and onto. Then there exist a* G_δ *subset* D *of* C, *a closed subset* F *of* C *and a dense* G_δ *subset* H *of* R *such that* $f: D \cap F \to H$ *is perfect onto where* $f(c) = (g \circ p)(c)$.
An elementary fact is:

Lemma. *Suppose that* $q: S \to T$ *is a minimal perfect (onto) map. If* N *is nowhere dense subset of* T *then* $q^{-1}(N)$ *is a nowhere dense subset of* S.

*) Institut für Mathematik, Johannes Kepler Universität, A-4040 Linz, Austria

Proof. If N is closed and $C = S \setminus (\operatorname{int} q^{-1}(N)) \neq S$ then $q(C)$ is a proper closed and dense subset of T which is impossible.

We do not have time to consider all multivalued mappings Φ; of course, we only consider

$$\Phi\colon C(K) \to \wp(K) \tag{1}$$

defined by

$$\Phi(x) = \{t \in K\colon x(t) = \sup_{s \in K} x(s) = \varrho x)\}. \tag{2}$$

Of course, $C(K)$ is the Banach algebra of continuous functions on the compact Hausdorff space K with the supremum norm. It is well known and quite easy to check that the projection from the graph $\{(x, k)\colon k \in \Phi(x)\}$ onto $C(K)$ is a minimal perfect mapping. It follows that for any open subset U of $C(K)$ the projection from

$$\{(x, k)\colon k \in \Phi(x) \text{ and } x \in U\}$$

onto U is also a minimal perfect mapping. Observe that for any open subset $W \subseteqq C(K)$ the set

$$\bigcup_{x \in W} \Phi(x)$$

is an open subset of K; if $\varepsilon > 0$ then

$$\{t\colon x(t) > \varrho(x) - \varepsilon\} \subseteqq \Phi(x \wedge (\varrho(x) - \varepsilon))$$

and $\|x - (x \wedge (\varrho(x) - \varepsilon))\| \leqq \varepsilon$. Observe, also, that if $N \subseteqq K$ is closed and nowhere dense then

$$\{x \in C(K)\colon \Phi(x) \cap N \neq \emptyset\}$$

is a closed and nowhere dense subset of $C(K)$.

Proposition. *Let K be a compact Hausdorff space and $W \neq \emptyset$ be an open subset of $C(K)$ and let $L \subseteqq C(K)$ be first category. Then $\Phi(W \setminus L)$ differs from the open set $\Phi(W)$ by a set of the first category and so $\Phi(W \setminus L)$ is a Baire property set.*

Proof. Let $L \subseteqq \bigcup_n N_n$ where each N_n is closed and nowhere dense and let $G = W \setminus \bigcup_n N_n$. Define

(i) $S = \{(x, k)\colon x \in W \text{ and } x(k) = \varrho(x)\}$;
(ii) $R = \{k\colon \text{there exists } x \in W \text{ such that } x(k) = \varrho(x)\}$ which is the projection of S;
(iii) g is the projection from S to R and
(iv) $T = \{(x, k)\colon k \in G \text{ and } x(k) = \varrho(x)\}$.

It is totally routine to show that $T \subseteq S$ is dense (the Lemma above) and is a Čech complete space (see [E]) and $g: S \to R$ is continuous, open and onto. Theorem II above says considerably more than that $\Phi(G)$ contains a dense G_δ subset of $\Phi(W)$.

Now, we go to the main result. Suppose that $E \in \mathbf{F}$. The first case to consider is that

$$\{x \in C(K): \Phi(x) \cap E \neq \emptyset\}$$

is first category (the assumption is that it is a Baire Property set). It follows from the Lemma above that

$$G = \{x \in C(K): \Phi(x) \cap E = \emptyset\}$$

contains a dense G_δ subset of $C(K)$. Hence, $\bigcup_{x \in G} \Phi(x)$ differs from K by a set of the first category (Theorem II). This proves that E is first category. Now, suppose that

$$\{x \in C(K): \Phi(x) \cap E \neq \emptyset\} = W \Delta N$$

where $W \neq \emptyset$ is an open set and N is first category. We have assumed that

$$N_1 = \{x \in C(K): \Phi(x) \cap E \neq \emptyset \text{ and } \Phi(x) \cap (T \setminus E) \neq \emptyset\}$$

is also a set of the first category. Let $\{P_n\}$ be a sequence of closed nowhere dense subsets of $C(K)$ such that

$$N \cup N_1 \cup (\bar{W} \setminus W) \subseteq \bigcup_n P_n = P.$$

Let $W_1 = C(K) \setminus \bar{W}$. We have that

$$\Phi(W \setminus P) \subseteq E,$$

$$\Phi(W_1 \setminus P) \cap E = \emptyset \text{ and}$$

$$K \setminus (\Phi(W \setminus P) \cup \Phi(W_1 \setminus P)) \text{ is first category}.$$

This proves that E is a Baire property set.

Theorem III. *Let K be a compact Hausdorff space and Φ defined as in (1) and (2). Then* $\mathbf{F}(\Phi)$ *is exactly the Baire property sets of K. In particular, E is in* $\mathbf{F}(\Phi)$ *if and only if E has the representation*

$$\Phi(W \setminus P) \subseteq E \subseteq \Phi(W) \cup N \subseteq \Phi(W \setminus P) \cup N$$

where W is an open subset of $C(K)$, P is first category in $C(K)$ and *N is first category in K.*

Corollary. *If K is a compact Hausdorff space then* $E \subseteqq K$ *is first category (respectively, E contains a dense* G_δ *subset of K) if and only if*

$\{x \in C(K): \Phi(x) \cap E \neq \emptyset\}$ *is first category*

(respectively, $\{x \in C(K): \Phi(x) \subseteqq E\}$ *contains a dense* G_δ *subset of* K).

Of course, we could play the same games for spaces K that are Baire spaces and Baire property sets in their Stone-Čech compactifications (see **[S1]**). The main result of **[S2]** combined with Theorem 6.11 of **[S1]** yields the following result.

Theorem. *Let* $A \subseteqq T \subseteqq K$ *where A is an α-favorable topological space dense in K, K is compact, T is a Baire property set in K and T is in a class of topological spaces introduced by us (see* **[S1]***). Then A contains a dense* G_δ *subspace homeomorphic to a complete metric space.*
If, in the result above, A is only a Baire space then A contains a dense G_δ subset that is metrizable. The following is how Theorem 8.12 of **[S1]** should be stated.

Theorem. *Let T be a compact and convex subset of some Hausdorff topological vector space and let E ben the extreme points of T. Suppose that there exists a space F such that* $E \subseteqq F \subseteqq \beta E$ *where F is a Baire property set in* βE *and is in a class of topological spaces introduced by us (see* **[S1]***). Then E contains a dense* G_δ *subset that is metrizable.*

Proof. Since E is α-favorable (a theorem of Choquet) it is a Baire space and it follows that F is a Baire space. As pointed out in **[S1]** this means that F contains a dense G_δ subset G of βE. It is very easy to check that G contains a dense G_δ set M that is completely metrizable; this is in **[S1]** and the topology is in **[E]**. Thus, $M \cap E$ is dense in E.

References

[E] ENGELKING, R. *General Topology,* PWN, Warsaw, 1977

[M] MICHAEL, E.

[S1] STEGALL, C. The Topology of Certain Spaces of Measures, Topology and its Applications 41 (1991), 73–112

[S2] STEGALL, C. Topological spaces with dense subspaces that are homeomorphic to complete metric spaces and the classification of $C(K)$ Banach spaces, Mathematika 34 (1987), 101–107