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One Counterexample Concerning the Fréchet Differentiability of Convex Functions on Closed Sets

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It is well known, that if F is a convex continuous function on a convex set G, then the Fréchet differentiability of F at some $x \in \text{int } G$ implies the norm-to-norm upper semicontinuity of ∂F at x. We consider the case of a convex Lipschitz function F defined on a closed convex subset K of a Banach space X, with the interior of K replaced by the set $N(K)$ of nonsupport points. We construct an example in l_2, which shows that in this case there can exist even a dense subset D of $N(K)$ such that F is Fréchet differentiable in every point of D and $\partial F : N(K) \to X^*$ is norm-to-norm upper semicontinuous at no point of D.

1 Introduction

We will consider a real valued function f defined on a closed nonvex subset K of a Banach space X. Differentiation properties of such a function f are usually examined in the case when the interior of K is nonempty. If the interior of K is empty it is possible to substitute it by the set $N(K)$ of so called nonsupport points of K.

Definition 1.1 *A point $x \in K$ is called a support point of K provided there exists a nonzero $x^* \in X^*$ such that*

$$\langle x^*, x\rangle = \sup\{\langle x^*, y\rangle;\ y \in K\}.$$

The set of all points in K which are not support points is denoted by N(K).

The set $N(K)$ has many properties similar to that of the interior of K. It is convex, and if $N(K) \neq \emptyset$, the $N(K)$ is dense in K (this is due to the fact, that if $x \in N(K)$ and $y \in K$, then $[x, y) \subset N(K)$). The separation theorem implies, that $N(K) = \text{int}K$ if the latter is nonempty. Also the set $N(K)$ is a G_δ subset of K [2], hence is a Baire space.

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Let us compare some basic differentiability properties of convex functions on open sets with the ones on the set $N(K)$.

Definition 1.2 *Let X be a Banach space, K be a closed convex subset of X, and f a convex function defined on K.*

(i) The subdifferential $\partial f(x)$ *of the convex function f at the point $x \in K$ is defined to be the set of all $x^* \in X^*$ satisfying*

$$\langle x^*, \ y - x\rangle \leqq f(y) - f(x) \ \text{for all} \ y \in K.$$

(ii) The function f is said to be Gâteaux differentiable at $x \in N(K)$ if $\partial f(x)$ is single valued.

(iii) The function f is Fréchet differentiable at $x \in N(K)$ if there exists a unique $x^* \in X^*$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ so that

$$0 \leqq f(y) - f(x) - \langle x^*, y - x\rangle \leqq \varepsilon\|y - x\|$$

for any $y \in N(K)$, $\|x - y\| < \delta$. We denote $f(x) := x^*$.

These definitions coincide with the usual ones when the interior of K is nonempty. It is well known that for a convex continuous function f defined on an open convex set G the subdifferential $\partial f(x)$ is nonempty for any $x \in G$. Moreover, the subdifferential mapping is norm-to-weak* upper semicontinuous. If we assume that f is locally Lipschitz at any point of $N(K)$, then we have the following:

Theorem 1.3 *Let X be a Banach space, K a closed convex subset of X, and f a convex function defined on K and locally Lipschitz at any point of $N(K)$. Then*

(i) (Verona [4]) *the subdifferential of f is nonempty at each point of $N(K)$.*

(ii) (Rainwater [3]) *Moreover the subdifferential mapping $\partial f: N(K) \to X^*$ is locally bounded and norm-to-weak* upper semicontinuous.*

For a convex continuous function f defined on an open convex set G the Gâteaux differentiability of f at a point $x \in G$ is equivalent to the existence of a selection for ∂f which is norm-to-weak* continuous at the point x. Similarly, the Fréchet differentiability of f at x is equivalent to the existence of a selection which is norm-to-norm continuous at x. Rainwater includes in [3] a proposition, which states, that such equivalences hold for a convex function which is locally Lipschitz at any point of $N(K)$. He really uses and proves there that the existence of a continuous selection implies differentiability and says that the other implication is straightforward. For the Gâteaux differentiability the equivalence really holds:

Proposition 1.4 [3] *Let X be a Banach space, K a closed convex subset of X. If f is convex on K and locally Lipschitz at any point of $N(K)$, then it is Gâteaux differentiable at a point $x \in N(K)$ iff there is a selection Φ for the subdifferential mapping $\partial f: N(K) \to X^*$ which is norm-to-weak* continuous at x.*

However, in the case of Fréchet differentiability we have only the following:

Proposition 1.5 [3] *Let X be a Banach space, K a closed convex subset of X. If f is convex on K, locally Lipschitz at any point of $N(K)$, and there is a selection Φ for the subdifferential mapping $\partial f: N(K) \to X^*$ which is norm-to-norm continuous at $x \in N(K)$, then f is Fréchet differentiable at the point x.*

As we will see in the Section 2 the other implication does not hold. There can even exist a dense subset D of $N(K)$ such that f is Fréchet differentiable at any point of D, but any selection for $\partial f: N(K) \to X^*$ is discontinuous at any point of D. However, due to the following theorem, in Asplund spaces the other implication holds on a dense G_δ subset of $N(K)$.

Theorem 1.6 [3] *Let X be an Asplund space, K a closed convex subset of X such that $N(K)$ is nonempty. If a function f is convex on K and locally Lipschitz at any point of $N(K)$, then there exists a dense G_δ subset G of $N(K)$ such that any selection for ∂f on $N(K)$ is norm-to-norm continuous at any point of G.*

2 Example

In this section we will construct a closed convex subset K of l_2, a convex Lipschitz function F on K, and a dense subset D of $N(K)$ such that F is Fréchet differentiable at any point of D, but any selection for $\partial f: N(K) \to X^*$ is discontinuous at any point of D.

Lemma 2.1 *Let X be a Banach space, K a closed convex subset of X. If functions f and g are convex on K and locally Lipschitz at any point of $N(K)$, then*

$$\partial(f+g)(x) = \partial f(x) + \partial g(x)$$

for any $x \in N(K)$.

Proof. The proof of this lemma is almost identical with Verona's proof of Theorem 1.3 (i), so let us only sketch it.

Define the cone K_x as the set of all $y \in X$ for which there exists some $t > 0$ so that $x + ty \in K$. If $x \in N(K)$, then K_x is dense in X. Now for $y \in K_x$ let

$$p_f(y) := \lim_{t \to 0+} \frac{1}{t}(f(x+ty) - f(x))$$

The function p_f is convex, uniformly continuous, and sublinear on the dense cone K_x. There is a unique convex continuous extension of p_f to all of X. Verona proves that $\partial f(x) = \partial p_f(0)$.

If we similarly define functions p_g, p_{f+g}, then by [1]

$$\partial p_{f+g}(0) = \partial p_f(0) + \partial p_g(0),$$

because p_f, p_g, p_{f+g} are convex and continuous on an open set (all of X). Therefore also $\partial(f+g)(x) = \partial f(x) + \partial g(x)$.

Remark 2.2 From Lemma 2.1 follows that if we have convex Lipschitz functions f, g on K and $x \in N(K)$ so that any selection for ∂f and any selection for ∂g on $N(K)$ are norm-to-norm continuous at x, then the same holds for $\partial(f+g)$.

Lemma 2.3 *Let X be a Banach space, K a closed convex subset of X. Let functions f, g, h: $K \to R$ be convex and locally Lipschitz at any point of $N(K)$ and Fréchet differentiable at some $x \in N(K)$, $c > 0$. Let*

(i) $f'(x) = 0$ and there exists a sequence $\{x_n\} \subset N(K)$ converging to x such that $\|y^*\| \geq c$ whenever $y^* \in \partial f(x_n)$.

(ii) any selection for ∂g on $N(K)$ is norm-to-norm continuous at x.

(iii) the function h is $(c/4) -$ Lipschitz.

Then the function $F = f + g + h$ is Fréchet differentiable at x, but no selection for ∂F on $N(K)$ is norm-to-norm continuous at x.

Proof. Clearly $F(x) = g'(x) + h'(x)$. By (ii) there exists $n_0 \in N$ so that for any $n > n_0$

$$\|g'(x) - y_g^*\| < c/4,$$

whenever $y_g^* \in \partial g(x_n)$.

Now let any $n > n_0$ be given and $y^* \in \partial F(x_n)$. By Lemma 2.1 there exist $y_f^* \in \partial f(x_n)$, $y_g^* \in \partial g(x_n)$, and $y_h^* \in \partial h(x_n)$ so that

$$y^* = y_f^* + y_g^* + y_h^*$$

Hence

$$\|F(x) - y^*\| \geq -\|g'(x) - y_g^*\| - \|h'(x)\| + \|y_f^*\| - \|y_h^*\| \geq$$
$$\geq -\frac{c}{4} - \frac{c}{4} + c - \frac{c}{4} = \frac{c}{4},$$

due to (iii). Consequently, no selection for ∂F on $N(K)$ is norm-to-norm continuous at x.

In the following we will consider a specific set $K := \{x = (a_1, a_2, \dots) \in l_2;\ 0 \leq a_n \leq 1/n\}$. The set K is convex and compact. If we define $K_0 := \{x = (a_1, a_2, \dots) \in l_2;\ 0 < a_n < 1/n\}$, then obviously $K_0 = N(K)$.

The following lemma shows that if any $z \in K_0$ is given, we can construct a Lipschitz convex function f_z on K, which is Fréchet differentiable at z, but there is no selection for ∂f on $N(K)$, which is norm-to-norm continous at z.

Lemma 2.4 *Let $z \in K_0$. Then there exists a nonnegative convex 2-Lipschitz function f_z: $K \to R$, and a sequence $\{z_k\} \subset N(K)$ converging to z, so that $|f_z| < 8$, $f_z'(z) = 0$ and $\|y^*\| \geq 1$ for any $y^* \in \partial f_z(z_k)$.*

Proof. Let $z = (\alpha_1, \alpha_2, \ldots) \in K_0$ be given. Choose an increasing sequence $\{n_k\}$ of natural numbers such that for any odd k

$$\frac{1}{4k}\left(\frac{1}{n_k} - \alpha_{n_k}\right) > \frac{1}{n_{k+1}}. \tag{1}$$

Denote by $\{e_n\}_{n=1}^{\infty}$ the orthonormal basis of l_2 and define (see Fig. 1)

$$x_k^* := 1/4k \; e_{n_k} + e_{n_{k+1}}$$

$$F_k(x) := \langle x_k^*, x - z\rangle - (1/4k)\,(1/n_k - \alpha_{n_k}) - 1/2\,(1/n_{k+1} - \alpha_{n_{k+1}}) \text{ and}$$

$$\Phi_k(x) := \begin{cases} \max\,(F_k, 0) & \text{for } k = 1, 3, 5, \ldots \\ 0 & \text{for } k = 2, 4, 6, \ldots \end{cases}$$

Let us define $f_z := \sup_{k \in N} \Phi_k$. Any of the functions Φ_k is convex, 2-Lipschitz and $\Phi_k(x) = 0$. Therefore the function f_z is convex and 2-Lipschitz, and $|f_z| < 8$.

Now let some odd k be fixed. We will establish some properties of the function Φ_k. First if $\|x - z\| < 1/(2n_{k+1})$ then due to (1)

$$F_k(x) \leqq \left\|\frac{1}{4k} e_{n_k} + e_{n_{k+1}}\right\| \left\|x - z\right\| - \frac{1}{n_{k+1}} < 0.$$

Consequently

$$\Phi_k = 0 \quad \text{on } B(z, 1/(2n_{k+1})). \tag{2}$$

Clearly also

$$\Phi_k(x) = 0 \text{ when } x - z \in \text{span}\{e_{n_m}, e_{n_{m+1}}\},\ m \neq k,\ m = 1, 3, 5, \ldots \tag{3}$$

Now let us take an arbitrary $x \in K$. If $F_k(x) < 0$ then $\Phi_k(x)/\|x - z\| = 0$. Now denote $(\beta_1, \beta_2, \ldots) = x - z$. If $F_k(x) > 0$ then

$$\frac{\beta_{n_k}}{4k} + \beta_{n_{k+1}} > \frac{1}{4k}\left(\frac{1}{n_k} - \alpha_{n_k}\right) + \frac{1}{2}\left(\frac{1}{n_{n_{k+1}}} - \alpha_{n_{k+1}}\right),$$

$$\beta_{n_k} \leqq \frac{1}{n_k} - \alpha_{n_k}, \qquad \beta_{n_{k+1}} \leqq \frac{1}{n_{k+1}} - \alpha_{n_{k+1}}.$$

Consequently due to (1)

$$\frac{\beta_{n_k}}{4k} \geqq \frac{1}{4k}\left(\frac{1}{n_k} - \alpha_{n_k}\right) + \frac{1}{2}\left(\frac{1}{n_{k+1}} - \alpha_{n_{k+1}}\right) - \beta_{n_{k+1}} \geqq$$

$$\geqq \frac{1}{4k}\left(\frac{1}{n_k} - \alpha_{n_k}\right) - \frac{1}{2}\left(1 n_{k+1} - \alpha_{n_{k+1}}\right) >$$

$$> \frac{1}{8k}\left(\frac{1}{n_k} - \alpha_{n_k}\right)$$

and we have that $\beta_{n_k} > \frac{1}{2}(1/n_k - \alpha_{n_k})$. Hence again due to (1)

$$\langle x_k^*, x - z\rangle = \frac{\beta_{n_k}}{4k} + \beta_{n_{k+1}} < \frac{1}{2k}(1/n_k - \alpha_{n_k}) < \frac{1}{k}\beta_{n_k} \leqq \frac{\|x - z\|}{k}.$$

Therefore

$$0 \leqq \frac{\Phi_k(x)}{\|x - z\|} = \frac{F_k(x)}{\|x - z\|} < \frac{\langle x_k^* \| x - z\rangle}{\|x - z\|} \leqq \frac{1}{k}.$$

Cousequently

$$0 \leqq \frac{\Phi_k(x)}{\|x - z\|} \leqq \frac{1}{k} \text{ for } x \in K. \tag{4}$$

The functions Φ_k and F_k equal on the halfspace $H := \{y;\ F_k(y) > 0\}$, so for any $y \in H$ we have that $\Phi_k'(y) = x_k^*$ and therefore $\langle \Phi_k'(y), e_{n_{k+1}}\rangle = 1$. If we define

$$u_k = z + \left(\frac{1}{n_k} - \alpha_{n_k}\right) e_{n_k} + \left(\frac{1}{n_{k+1}} - \alpha_{n_{k+1}}\right) e_{n_{k+1}},$$

then $u_k \in H \cap K$, and the sequence $\{u_k\}$ converges to z. Choose any $z_k \in K_0 \cap H \cap B(u_k, 1/k)$. Then $\{z_k\}$ also converges to z.

Now let us go back to the function f_z and prove that $f_z'(z) = 0$.

Let $\varepsilon > 0$ be given and an odd k is such that $1/k < \varepsilon$. Then for $x \in B(z, 1/2n_{k+1})$ we have

$$0 \leqq \frac{f_z(x) - f_z(z)}{\|x - z\|} = \frac{\sup_{m \in N} \Phi_m(x)}{\|x - z\|}.$$

Due to (2) we have $\Phi_1(x) = \ldots = \Phi_k(x) = 0$, hence using (4) we get

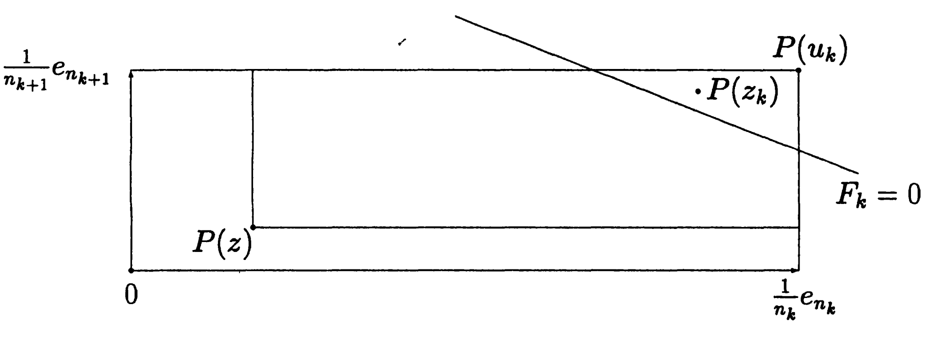

Fig. 1 The situation in the subspace span $\{e_{n_k}, e_{n_{k+1}}\}$ (P denotes the projection on this subspace).

$$0 \leqq \frac{f_z(x) - f_z(z)}{\|x - z\|} = \sup_{m>k} \frac{\Phi_m(x)}{\|x - z\|} < \frac{1}{k} < \varepsilon .$$

Due to (3) we have $f_z(x) = \Phi_k(x)$ if $x - z \in \operatorname{span}\{e_{n_k}, e_{n_{k+1}}\}$ for $k = 1, 3, 5,\ldots$ Therefore for any $y^* \in \partial f_z(z_k)$ we have

$$\langle y^*, e_{n_{k+1}}\rangle = \langle \Phi_k'(z_k), e_{n_{k+1}}\rangle = 1,$$

hence $\|y^*\| \geqq 1$.

Now let us show that the set of points where the Fréchet differentiability is not equivalent to the existence of a continuous selection can be a dense subset of $N(K)$.

Example 2.5 *There exists a compact convex subset K of l_2, convex Lipschitz function F on K, and a dense subset D of $N(K)$ such that for any $x \in D$ the function F is Fréchet differentiable at x, but no selection for ∂F on $N(K)$ is norm-to-norm continuous at x.*

Proof. Let the sets K and $K_0 = N(K)$ be as above, and $\{y_n\}$ be a sequence of points dense in K. By induction, we will construct a dense subset $D = \{x_1, x_2,\ldots\}$ of K_0 and a sequence of 4-Lipschitz convex functions $\{f_i\}$ on K so that $|f_i| < 8$ and the following conditions hold ($i = 1, 2,\ldots$):

(i) $f_i'(x_i) = 0$

(ii) there exists a sequence $\{x_i^k\} \subset N(K)$ converging to x_i, such that $\|y^*\| \geqq 1$ whenever $y^* \in \partial f_i(x_i^k)$.

(iii) any selection for $\partial f_i : N(K) \to X^*$ is norm-to-norm continuous at any point x_j, $j \neq i$, $j = 1, 2,\ldots$

In the first step, let x_1 be any point in K_0 and $f_1 := f_{x_1}$, where f_{x_1} is the convex 2-Lipschitz function from Lemma 2.4.

The nth step: Until now finitely many points $x_1,\ldots, x_{n-1}$ and 4-Lipschitz convex functions $f_1,\ldots, f_{n-1}$ have been constructed so that (i) and (ii) holds for $i = 1,\ldots, n - 1$. Moreover, any selection for $\partial f_i : N(K) \to X^*$ is norm-to-norm continuous at any point x_j, $i \neq j$, $i, j = 1, 2,\ldots, n - 1$ (i.e. the statement (iii) holds for functions and points constructed until now).

Let G_i (for $i = 1,\ldots, n - 1$) be the dense G_δ subset of $N(K)$ provided for the function f_i by the Theorem 1.6. Then the set $H_n := \bigcap_{i=1}^{n-1} G_i$ is also dense and G_δ, because $N(K)$ is a Baire space. Now choose some point x_n in the set $B(y_n, 1/n) \cap H_n$

Denote $d := \min_{i=1,\ldots,n-1} \|x_i - z_n\|$. Because $x_n \in H_n$ we have $d > 0$. Define a convex 4-Lipschitz function $\psi(y) := 4\|x_n - y\| - d$.

Let f_{x_n} be the function from Lemma 2.4. Define

$$f_n := \max \{f_{x_n}, \psi\} .$$

The function f_n is 4-Lipschitz, and because $f_{x_n} \geqq 0$ we have

$$f_n = f_{x_n} \text{ on } B(x_n, d/4) \cap K.$$

Therefore (i) and (ii) are satisfied for $i = n$.

Because f_{x_n} is 2-Lipschitz,

$$f_n = \psi \text{ on } K - B\left(x_n, \frac{d}{2}\right),$$

Due to this and the fact that $x_n \in H_n$, the statement (iii) holds for functions and points constructed until now.

Clearly, the constructed sequence of points $D = \{x_1, x_2, \ldots\}$ is dense in $N(K)$ and the sequences D and $\{f_1, f_2, \ldots\}$ satisfy (i), (ii), and (iii).

Now define

$$F := \sum_{i=1}^{\infty} \frac{1}{2^i} f_i.$$

Any of the functions f_i is 4-Lipschitz, convex, and $|f_i| < 8$. Therefore F is Lipschitz and convex on K. We will show, that F has also the other required properties.

Let $n \in N$ be given. Denote

$$c := \frac{1}{2^n}$$

$$f := \frac{1}{2^n} f_n, \quad g := -f + \sum_{i=1}^{n+3} \frac{1}{2^i} f_i,$$

$$h := \sum_{i=n+4}^{\infty} \frac{1}{2^i} f_i$$

The functions f, g, h are Fréchet differentiable at x_n. This fact is trivial for f and g, and can be easily proved from definition for h. Due to the Remark 2.2 any selection for ∂g on $N(K)$ is norm-to-norm continuous at x_n. Because the functions f, g, and h obviously satisfy also the other properties required in Lemma 2.3 (with $x = x_n$), the function $F = f + g + h$ is Fréchet differentiable at x_n, but no selection for $N(K)$ is norm-to-norm continuous at x_n.

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