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The Indexed Open Covering Theorem

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Abstract. The main result of this note is a theorem on an open covering of a Tychonoff cube. There are some results related to the following question: under what conditions does $f: I^S \to \mathbb{R}^T$ map two opposite faces of the cube I^S onto disjoint sets?

§1. An open covering theorem. Let S and T be non-empty sets. The symbol I^S denotes the Tychonoff cube,

$$I^S := \{x\colon S \to [-1, 1] \mid x \text{ is a map}\}$$

and $\mathbb{R}^T$ the product of T copies of the real line $\mathbb{R}$,

$$\mathbb{R}^T := \{x\colon T \to \mathbb{R} \mid x \text{ is a map}\}$$

Both sets I^S and $\mathbb{R}^T$ are equipped with the Cartesian product topology. For each $s \in S$ let us denote

$$I_s^- := \{x \in I^S\colon x_s = -1\}, \qquad I_s^+ := \{x \in I^S\colon x_s = 1\}$$

the s-opposite faces of the cube I^S. The symbol ∂I^S denotes the pseudoboundary of the cube I^S,

$$\partial I^S := \bigcup\{I_s^- \cup I_s^+ : s \in S\}$$

Sometimes, when a set S or T is finite, $|S| = n$ or $|T| = m$, we shall use symbols I^n, $\mathbb{R}^m$ instead of I^S, $\mathbb{R}^T$.

In this note the following result plays a central role.

The Indexed Open Covering Theorem. *Let $\{U_s\colon s \in S\}$ be an open covering of the cube I^S. Then there exist an index $s \in S$ and a connected subset $U \subset U_s$ such that $I_s^- \cap U \neq \emptyset \neq I_s^+ \cap U$.*

In order to demontrate an importance of this theorem let us prove.

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The Bohl-Brouwer Fixed Point Theorem. *Any continuous map* $f\colon I^S \to I^S$ *has a fixed point.*

Proof. Let $f\colon I^S \to I^S$ be a continuous map and suppose that $f(x) \neq x$ for each $x \in I^S$. For each $s \in S$, let

$$U_s := \{x \in I^S : f_s(x) \neq x_s\}.$$

From the supposition it follows that the family $\{U_s : s \in S\}$ is a covering of I^S. According to the Indexed Open Covering Theorem there exist an index $s \in S$, a connected set $U \subset U_s$, and points $a, b \in U$ such that $a_s = -1$ and $b_s = 1$. For each $s \in S$, $f_s(x) \in [-1, 1]$ and so

$$f_s(a) - a_s = 1 + f_s(a) \geqq 0 \quad \& \quad f_s(b) - b_s = f_s(b) - 1 \leqq 0.$$

Since u is connected there is a point $c \in U$ such that $f_s(c) - c_s = 0$. From the definition of the set U_s we have $c \notin U_s$, a contradiction with $c \in U \subset U_s$. The proof is complete.

It will be shown in section 4 that the Indexed Open Covering Theorem is equivalent to the Bohl-Brouwer Theorem.

§2. Cardinal dimension. Let τ be a cardinal number finite or infinite. A normal space X is said to be of cardinal dimension greater than or equal to τ, $\operatorname{dc} X \geqq \tau$, provided that there exists a family $\{\langle A_s, B_s\rangle : s \in S\}$, $|S| \geqq \tau$, of pairs of non-empty disjoint closed sets i.e., $A_s \cap B_s = \emptyset$ for all $s \in S$ such that for every open covering $\{U_s : s \in S\}$ of X there exists an index $s \in S$ and a connected set $U \subset U_s$ such that $A_s \cap U \neq \emptyset \neq B_s \cap U$.

A normal space X is said to be of dimension τ, $\operatorname{dc} X = \tau$, provided that $\operatorname{dc} X \geqq \tau$ and the inequality $\operatorname{dc} X \geqq \eta$ does not hold for any $\eta > \tau$.

In the definition of cardinal dimension it is possible that $S = \emptyset$. Thus for each normal space X we have $\operatorname{dc} X \geqq 0$. On the other hand, the definition does not guarantee that for every normal space X there exists a cardinal number τ such that $\operatorname{dc} X = \tau$. We shall prove

Theorem 1. $\operatorname{dc} I^S = |S|$.

To prove this theorem we need two lemmas.

Lemma 1. *Let* X *be a normal space with* $\operatorname{dc} X \geqq \tau$. *Then there exist a continuous map* $f\colon X \xrightarrow{\text{onto}} I^S$, $|S| = \tau$, *and a set* $A \subset X$ *such that* $f(A) \subset \partial I^S$ *and for any continuous map* $g\colon X \to \mathbb{R}^S$, $g \mid A = f \mid A$ *implies* $I^S \subset g(X)$. *Moreover, if* S *is finite then* A *is a closed set.*

Proof. Let $\{\langle A_s, B_s\rangle : s \in S\}$, $|S| = \tau$, be a family of pairs of non-empty disjoint closed sets satisfying the conditions of the definition of cardinal dimension. Define

$$A := \bigcup\{A_s \cup B_s : s \in S\}.$$

Since X is normal, there exists a continuous map $f\colon X \to I^S$ having the following property: for each $s \in S$ and for each $x \in X$

$$x \in A_s \Rightarrow f_s(x) = -1 \quad \& \quad x \in B_s \Rightarrow f_s(x) = 1\,.$$

It is clear that $f(A) \subset \partial I^S$. Let $g\colon X \to \mathbb{R}^S$ be a continuous map such that $g \mid A = f \mid A$. We shall show that $I^S \subset g(X)$. Suppose that there is a point $c \in I^S \setminus g(X)$. For each $s \in S$ let

$$U_s := \{x \in X\colon g_s(x) \neq c_s\}\,.$$

The supposition implies that the family $\{U_s\colon s \in S\}$ is an open covering of X. Coose an index $s \in S$, a connected set $U \subset U_s$ and points $a \in U \cap A_s$, $b \in U \cap B_s$. Since $g_s(a) - c_s \leqq 0$ and $g_s(b) - c_s \geqq 0$ we infer that there is a point $d \in U$ such that $g_s(d) - c_s = 0$. Then $d \notin U_s$, a contradiction with $d \in U \subset U_s$.

Lemma 2. *Let $X \subset \mathbb{R}^n$ be a compact boundary subspace. Then for each continuous map $f\colon A \to \mathbb{R}^n \setminus \{0\}$, where A is a closed subset of X, there exists a continuous map $F\colon X \to \mathbb{R}^n \setminus \{0\}$ such that $F \mid A = f$.*

Proof. (I). First, we shall show that if $X \subset \mathbb{R}^n$ is a compact boundary set then for each map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of class C^1, the image $f(X)$ is a boundary subset.

The Sard Lemma (cf. Deimling [4]) asserts that the set $f(D)$, where

$$D := \{x \in \mathbb{R}^n\colon \det f'(x) = 0\}\,,$$

has n-dimensional Lebesgue measure equal to zero. Thus the set $f(X \cap D)$, as a compact set of measure zero, is of first category. From the Inverse Function Theorem (cf. [4]) for each $x \in \mathbb{R}^n \setminus D$ there exists an open set $U \subset \mathbb{R}^n$, $x \in U$, such that $f \mid U\colon U \to f(U)$ is a homeomorphism onto the open subset $f(U)$ of $\mathbb{R}^n$. Thus $f(X \cap U)$ is also a set of first category. Since the space $\mathbb{R}^n$ has a countable base, it is easy to observe that $f(X)$ is a set of first category. From the Baire Category Theorem we infer that $\operatorname{Int} f(X) = \emptyset$.

(II). Let us proceed to the proof. Since $f(A)$ is a closed subset and $0 \notin f(A)$, there exists an $\varepsilon > 0$ such that $B(0, \varepsilon) \cap f(A) = \emptyset$, where $B(a, \varepsilon) := \{x \in \mathbb{R}^n\colon \|x - a\| < \varepsilon\}$. According to the Stone-Weierstrass Theorem there exists a map $f_1\colon \mathbb{R}^n \to \mathbb{R}^n$ of class C^1 such that $\|f(x) - f_1(x)\| < \frac{\varepsilon}{8}$ for each $x \in A$. The set $f_1(X)$ has empty interior and therefore there exists a point $a \notin f_1(X)$ such that $0 < \|a\| < \frac{\varepsilon}{8}$. We have $f_1(x) - a \neq 0$ for each $x \in X$. Let us put $g(x) := f_1(x) - a$. Then for each $x \in A$,

$$\|f(x) - g(x)\| < \frac{\varepsilon}{4} \quad \& \quad \|g(x)\| > \frac{\varepsilon}{2} \tag{1}$$

Define a continuous function $r\colon X \to \mathbb{R}$

$$(2) \qquad r(x) := \max\left\{\|g(x)\|, \frac{\varepsilon}{2}\right\}.$$

From (1) we get

$$(3) \qquad r(x) = \|g(x)\| \qquad \text{for each} \quad x \in A.$$

Now let us put for each $x \in X$,

$$(4) \qquad G(x) := r(x)\,\frac{g(x)}{\|g(x)\|}.$$

From (2)−(4) we get

$$(5) \qquad G \mid A = g \mid A \quad \& \quad \|g\| \geq \frac{\varepsilon}{2}.$$

Let $h(x) := f(x) - G(x)$, for $x \in A$. From (5) and (1) we have, $\|h(x)\| < \frac{\varepsilon}{4}$ for each $x \in A$. In view of the Tietze-Urysohn Theorem the map $h\colon A \to B(0, \frac{\varepsilon}{4})$ has a continuous extension $H\colon X \to B(0, \frac{\varepsilon}{4})$. Now, we can define a continuous extension $F\colon X \to \mathbb{R}^n \setminus \{0\}$ of the map f,

$$(6) \qquad F(x) := H(x) + G(x), \qquad x \in X.$$

We have $F \mid A = f$ and one can verify that $\|F\| \geq \frac{\varepsilon}{4}$ because $\|G\| \geq \frac{\varepsilon}{2}$ and $\|H\| \leq \frac{\varepsilon}{4}$ yield, $\|F(x)\| \geq \|G(x)\| - \|H(x)\| \geq \frac{\varepsilon}{2} - \frac{\varepsilon}{4}$. This completes the proof.

Proof of Theorem 1. The Indexed Open Covering Theorem implies that $\operatorname{dc} I^S \geq |S|$. To prove the equality $\operatorname{dc} I^S = |S|$ we must show that there is no cardinal number $\tau > |S|$ such that $\operatorname{dc} I^S \geq \tau$. Suppose that such a number τ exists. Consider two cases:

(I). $|S|$ is infinite. From Lemma 1 there exists a continuous map $f\colon I^S \xrightarrow{\text{onto}} I^T$, $|T| = \tau$. But then (cf. [5], Chapter 3)

$$|T| = \text{weight } I^T \leq \text{weight } f(I^S) \leq \text{weight } I^S = |S|,$$

a contradiction with $|S| < |T|$.

(II). $|S|$ is finite. Let $|S| = n$. Without loss of generality we may assume that $\tau = n + 1$ and I^S is a compact boundary subspace of I^T, where $|T| = n + 1$. From Lemma 1 it follows that there exist a closed subset A of I^S and a continuous map $f\colon A \to \mathbb{R}^T \setminus \{0\}$ such that for each continuous extension $F\colon I^S \to \mathbb{R}^T$ of f we have $I^T \subset F(I^S)$. But this contradicts Lemma 2.

§3. On preserving of disjoint faces. In this section we need the following.

Lemma 3. *Let $f\colon I^S \to X$ be a continuous map into a normal space X. Then* $\operatorname{dc} X \geq |T|$, *where* $T := \{s \in S\colon f(I_s^-) \cap f(I_s^+) = \emptyset\}$.

Proof. For each $t \in T$ let us put $A_t := f(I_t^-)$ and $B_t := f(I_t^+)$. We shall verify that the family $\{\langle A_t, B_t\rangle : t \in T\}$ of pairs of non-empty disjoint closed sets, realizes the definition of cardinal dimension. Let $\{U_t : t \in T\}$ be an open covering of X. For each $s \in S$, set $W_s := f^{-1}(U_s)$ for $s \in T$, and $W_s = \emptyset$ for $s \in S \setminus T$. The family $\{W_s : s \in S\}$ is an open covering of I^S. According to the Indexed Open Covering Theorem there exist an index $s \in T \subset S$ and a connected set $W \subset W_s$ such that $I_s^- \cap W \neq \emptyset \neq I_s^+ \cap W$. It is clear that $f(W)$ is connected and $A_s \cap f(W) \neq$ $\neq 0 \neq B_s \cap f(W)$. This completes the proof.

Theorem 2. *For each continuous map* $f: I^S \to \mathbb{R}^T$ *the following inequality holds:* $|\{s \in S: f(I_s^-) \cap f(I_s^+) = \emptyset\}| \leq |T|$.

Proof. Let $i: \mathbb{R}^T \to I^T$ be a topological embedding. It is clear that

$$f(I_s^-) \cap f(I_s^+) = \emptyset \Leftrightarrow (i \circ f)(I_s^-) \cap (i \circ f)(I_s^+) = \emptyset .$$

Thus without loss of generality we may assume that $f(I^S) \subset I^T$. Observe that according to Lemma 3 the inequality

$$|\{s \in S: f(I_s^-) \cap f(I_s^+) = \emptyset\}| > |T|$$

implies that $\operatorname{dc} I^T \geq \tau > |T|$, a contradiction with Theorem 1. Thus Theorem 2 is proved.

Theorem 3. *Assume that* $f: I^S \to \mathbb{R}^T$ *is a continuous map. If* $0 < |T| < \infty$ *and* $|\{s \in S: f(I_s^-) \cap f(I_s^+) = \emptyset\}| = |T|$, *then the interior of the set* $f(I^S)$ *is non-empty.*

Proof. Set $X := f(I^S)$. From Lemma 3 we infer that $\operatorname{dc} X \geq |T|$. Suppose that the subspace $X \subset \mathbb{R}^T$ has empty interior. Then comparing Lemma 2 with Lemma 1 we get a contradiction.

Theorem 4. *Let* $0 < |S| < \infty$. *Assume that* $f: I^S \to \mathbb{R}^S$ *is a continuous map such that* $f(I_s^-) \cap f(I_s^+) = \emptyset$ *for each* $s \in S$. *Then the set* $f(\partial I^S)$ *separates the Euclidean space* $\mathbb{R}^S$.

Proof. Set $A := f(\partial I^S)$, $X := f(I^S)$. Repeating the proof of Lemma 1 we infer that there exists a continuous map $g_1: A \to \partial I^S$ such that for each continuous map $g: X \to I^S$, $g_1 = g \mid A$ implies $g(X) = I^S$. But according to the Alexandroff-Borsuk Separation Theorem (cf. Borsuk [2] or Alexandroff and Pasynkov [1], Chapter 8), the set A separates $\mathbb{R}^n$.

§4. The Poincarè Theorem and its equivalent formulations. In this section let us fix a natural number $n \geq 1$ and denote by

$$d(x, A) := \inf\{\|x - a\| : a \in A\}$$

the distance between the point x and the set A.

Theorem 5. *The following statements are equivalent:*

(i) *(the Poincarè-Miranda Theorem). Let* $f\colon I^n \to \mathbb{R}^n$ *be a continuous map such that* $f_i(I_i^-) \subset (-\infty, 0]$ & $f_i(I_i^+) \subset [0, \infty)$ *for each* $i \leqq n$. *Then there exists a point* $c \in I^n$ *such that* $f(c) = 0$.

(ii) *If pairs* $\langle F_i^-, F_i^+\rangle$, $i = 1, \ldots, n$, *of closed sets are such that* $I^n = F_i^- \cup F_i^+$ *and* $I_i^- \subset F_i^-$, $I_i^+ \subset F_i^+$ *for each* $i \leqq n$, *then the intersection* $\bigcap\{F_i^- \cap F_i^-\colon i \leqq n\}$ *is non-empty.*

(iii) *If a family* $\{U_1, \ldots, U_n\}$ *is an open covering of* I^n *then there exist: an index* $i \leqq n$ *and a connected set* $U \subset U_i$ *such that* $I_i^- \cap U \neq \emptyset \neq I_i^+ \cap U$.

(iv) *(the Bohl-Brouwer Theorem). Any continuous map* $f\colon I^n \to I^n$ *has a fixed point.*

Proof. (i) ⇒ (ii). For each $i \leqq n$ let us define

$$f_i(x) := d(x, F_i^-) - d(x, F_i^+), \qquad x \in I^n$$

Since $I_i^- \subset F_i^-$ and $I_i^+ \subset F_i^+$, we have for each $i \leqq n$

$$f_i(I_i^-) \subset (-\infty, 0] \quad \& \quad f_i(I_i^+) \subset [0, \infty).$$

According to (i) there is a point $c \in I^n$ such that $f(c) = 0$, where $f = (f_1, \ldots, f_n)\colon I^n \to \mathbb{R}^n$. This means that for each $i \leqq n$,

$$d(c, F_i^-) = d(c, F_i^+)$$

But $c \in F_i^- \cup F_i^+$. Thus the following condition holds for each $i \leqq n$,

$$d(c, F_i^-) = 0 = d(c, F_i^+).$$

Since the sets F_i are closed, the above equalities are equivalent to

$$c \in \bigcap\{F_i^- \cap F_i^+ : i \leqq n\}.$$

(ii) ⇒ (iii). Suppose that (iii) does not hold. For each $i \leqq n$ let us define

$$U_i^- := \bigcup\{U \subset U_i\colon U \cap I_i^- \neq \emptyset,\ U \text{ is a connected component of } U_i\},$$
$$U_i^+ := U_i \setminus U_i^-.$$

The sets U_i are open and $U_i^- \cap U_i^+ = \emptyset$. Denote by

$$F_i^- := I^n \setminus U_i^+, \qquad F_i^+ := I^n \setminus U_i^-.$$

From the supposition we get

$$I_i^- \subset F_i^-, \ I_i^+ \subset F_i^+ \quad \& \quad I^n = F_i^- \cup F_i^+.$$

From (ii) there is a point $c \in \bigcap\{F_i^- \cap F_i^+ : i \leqq n\}$. But $\bigcap\{F_i^- \cap F_i^+ : i \leqq n\} = I^n \setminus \bigcup\{U_i\colon i \leqq n\}$ implies that the family $\{U_i\colon i \leqq n\}$ is not a covering of I^n, a contradiction.

(iii) ⇒ (iv). A proof of this implication is given in §1.
(iv) ⇒ (i). Let $f\colon I^n \to \mathbb{R}^n$ be a continuous map such that

$$f_i(I_i^-) \subset (-\infty, 0] \quad \& \quad f_i(I_i^+) \subset [0, \infty)\,.$$

For each $i \leqq n$ let us put

$$F_i^- := f_i^{-1}(-\infty, 0]\,, \qquad F_i^+ := f_i^{-1}[0, \infty)\,.$$

Now, define a continuous map $g\colon I^n \to \mathbb{R}^n$, $g = (g_1, \ldots, g_n)$,

$$g_i(x) := x_i - d(x, F_i^-) + d(x, F_i^+)\,.$$

Since $d(x, I_i^-) = 1 + x_i$, $d(x, I_i^+) = 1 - x_i$ and $I_i^\varepsilon \subset F_i^\varepsilon$, we get

$$-1 = x_i - d(x, I_i^-) \leqq g_i(x) \leqq x_i + d(x, I_i^+) = 1\,.$$

In consequence we infer that $g(I^n) \subset I^n$. From (iv) there is a point $c \in I^n$ such that $g(c) = c$. But this implies that for each $i \leqq n$,

$$d(c, F_i^-) = d(c, F_i^+)$$

and $c \in F_i^- \cup F_i^+$ yields for each $i \leqq n$,

$$d(c, F_i^-) = 0 = d(c, F_i^+)$$

or equivalently $c \in \bigcap\{F_i^- \cap F_i^+ : i \leqq n\}$ and so $f(c) = 0$.

The statement (i) of Theorem 5 was announced by Poincarè [6] in 1883 and rediscovered in 1940 by Miranda, who showed that it was equivalent to the Brouwer Fixed Point Theorem (cf. Browder [3]).

Proof of the Indexed Open Covering Theorem. Let $\{U_s\colon s \in S\}$ be an open covering of the cube I^S. Since I^S is a compact space there exists a finite set $\{s_1, \ldots, s_n\} \subset S$ such that

$$I^S = U_{s_1} \cup \ldots \cup U_{s_n}\,.$$

Let us put $I^n := I_{s_1} \times \ldots \times I_{s_n}$. Let $h\colon I^n \to I^S$ be a continuous map such that for each $i \leqq n$,

$$h(I_i^-) \subset I_{s_i}^- \quad \& \quad h(I_i^+) \subset I_{s_i}^+\,.$$

For example: let $h_s(x) = 0$ if $s \in S \setminus \{s_1, \ldots, s_n\}$ and $h(x) = x_s$ if $s \in \{s_1, \ldots, s_n\}$. Finally, let $W_i := h^{-1}(U_{s_1})$ for $i = 1, \ldots, n$. From Theorem 5 (iii) it follows that there exist: an index $i \leqq n$ and a connected set $W \subset W_i$ such that $I_i^- \cap W \neq \neq \emptyset \neq I_i^+ \cap W$. It is clear that if $U_i = h(W_i)$, $U = h(W)$ and $s = s_i$ then $I_s^- \cap U \neq \emptyset \neq I_s^+ \cap U$.

We conclude this paper with a remark which enables us to estimate the significance of the Bohl-Brouwer Theorem in dimension theory:

If there exists a normal space X such that dc $X \geqq \tau$, then each continuous map $f: I^S \to I^S$, $|S| = \tau$, has a fixed point.

We leave an easy proof of this remark to the reader.

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