Edward Grzegorek

On extensions of $\sigma$-fields of sets

*Acta Universitatis Carolinae. Mathematica et Physica*, Vol. 34 (1993), No. 2, 47--49

Persistent URL: `http://dml.cz/dmlcz/701992`

1993 ACTA UNIVERSITATIS CAROLINAE—MATHEMATICA ET PHYSICA VOL. 34. NO. 2

# On Extensions of $\sigma$-Fields of Sets

E. GRZEGOREK

Gdańsk*)

*Received 14 April 1993*

We assume Zermelo-Fraenkel set theory with the axiom of choice. The letter $\lambda$ will denote arbitrary cardinal while $\omega$ denotes the first infinite cardinal. If $X$ is a set then $|X|$ denotes the cardinality of $X, \mathscr{P}(X)$ is the power set of $X$,

$$[X]^{\leq\lambda} = \{Y \subseteq X : |Y| \leq \lambda\},$$
$$[X]^{<\lambda} = \{Y \subseteq X : |Y| < \lambda\} \quad \text{and}$$
$$[X]^{\lambda} = \{Y \subseteq X : |Y| = \lambda\}.$$

If $\mathscr{C} \subseteq \mathscr{P}(X)$ then: $\mathscr{C}$ is a partition of $X$ if $\bigcup \mathscr{C} = X, \emptyset \notin \mathscr{C}$ and elements of $\mathscr{C}$ are pairwise disjoint; a set $S$ is a selector of $\mathscr{C}$ if $S \subseteq \bigcup \mathscr{C}$ and $|S \cap Y| = 1$ for every $Y \in \mathscr{C}$. A family $\mathscr{C} \subseteq \mathscr{P}(X)$ is proper if $\mathscr{C} \neq \mathscr{P}(X)$. An ideal $\mathscr{I}$ on a set $X$ is a collection of subsets of $X$ that is closed under subset formation and finite unions. A family $\mathscr{F} \subseteq \mathscr{P}(X)$ is called a filter on $X$ if the family $\mathscr{I} = \{X \setminus F : F \in \mathscr{F}\}$ is an ideal on $X$. $\mathscr{I}$ and $\mathscr{F}$ are called then mutually dual. A filter $\mathscr{F}$ on $X$ is called uniform if $|F| = |X|$ for every $F \in \mathscr{F}$. An ideal $\mathscr{I}$ on $X$ is called uniform if the dual filter to $\mathscr{I}$ on $X$ is uniform. A family $\mathscr{A} \subseteq [X]^{|X|}$ is called a pseudobasis for a filter $\mathscr{F}$ on $X$ if $|\mathscr{A}| \leq |X|$ and for every $F \in \mathscr{F}$ there is some $A \in \mathscr{A}$ such that $A \subseteq F$ (see [1]). A filter is called $\sigma$-filter if it is closed under countable intersections. A $\sigma$-ideal is an ideal which is closed under countable unions. Let $R$ be the real line.

A version of the following theorem was proved in [1] in order to obtain a negative answer to a problem of Ulam (problem 34 in [4] and modified versions of it in [3] p. 15 and [5] p. 314).

**Theorem A.** *(Grzegorek and Węglorz [1] p. 286 and p. 289). There exists a proper $\sigma$-field $\mathscr{A}$ of subsets of $R$ such that:*

*(a) all Lebesgue measurable subsets of $R$ are in $\mathscr{A}$.*

*(b) for every partition $\mathscr{C} \subseteq [R]^{\leq\omega}$ of $R$ there is a selector of $\mathscr{C}$ in $\mathscr{A}$.*

*(c) $[R]^{<2^\omega} \subseteq \mathscr{A}$.*

*) Institute of Mathematics, University of Gdańsk, Wita Stwosza 57, 80-952 Gdańsk, Poland

The above Theorem A was formulated in "Added in proof" in [1]. In fact the $\sigma$-field constructed in [1] has all required in Theorem A properties (see our theorem 1.1 in [6]). The aim of the present note is the following generalisation of Theorem A.

**Theorem B.** *(i) Let $\mathcal{F}$ be a uniform $\sigma$-filter on $R$ and let $\mathcal{B}$ be a $\sigma$-field of subsets of $R$ such that $|\mathcal{B}| \leqq 2^\omega$. Then there exists a proper $\sigma$-field $\mathcal{A}$ on $R$ such that:*

*(a) the $\sigma$-field generated by $\mathcal{B}$ and $\mathcal{F}$ is contained in $\mathcal{A}$,*

*(b) for every partition $\mathcal{C} \subseteqq [R]^{\leq\omega}$ of $R$ there is a selector of $\mathcal{C}$ in $\mathcal{A}$,*

*(c) $[R]^{<2^\omega} \subseteqq \mathcal{A}$.*

*(ii) If $2^\lambda \leqq 2^\omega$ for all $\lambda < 2^\omega$ then in (i) instead of "$|\mathcal{B}| \leqq 2^\omega$" we can assume only that "$|\mathcal{B}| < 2^{2^\omega}$".*

**Remark 1.** Assuming additionally that the $\sigma$-filter $\mathcal{F}$ has a pseudobasis Theorem B can be easily obtained from [1] (compare our remarks after Theorem A). To see how Theorem A follows from Theorem B put $\mathcal{B}$ = Borel $\sigma$-field on $R$ and $\mathcal{F}$ = the filter dual to the ideal of the sets of the Lebesgue measure zero.

**Proof of Theorem B.** Let $\mathcal{F}$ and $\mathcal{B}$ satisfy the assumptions of Theorem B. Since $\mathcal{F}$ is uniform there exists, by a result of Węglorz [6], a uniform $\sigma$-filter $\mathcal{G}$ on $R$ such that $\mathcal{F} \subseteqq \mathcal{G}$, $[R]^{<2^\omega} \subseteqq \mathcal{I}$, where $\mathcal{I}$ is the ideal dual to the filter $\mathcal{G}$, and for every partition $\mathcal{C} \subseteqq [R]^{\leq\omega}$ of $R$ there is a selector of $\mathcal{C}$ in $\mathcal{G}$. Let $\mathcal{A}$ be the $\sigma$-field generated by $\mathcal{B}$ and $\mathcal{G}$. It is evident that $\mathcal{A}$ satisfies (a), (b) and (c). It remains to prove that $\mathcal{A}$ is proper. Suppose not. Then $\mathcal{B} \triangle \mathcal{I} = \mathcal{P}(R)$, where

$$\mathcal{B} \triangle \mathcal{I} = \{(B \setminus X) \cup (X \setminus B) \colon B \in \mathcal{B} \text{ and } X \in \mathcal{I}\}.$$

Consider the Boolean algebra $\mathcal{P}(R)/\mathcal{I}$ the set of all equivalence classes of subsets of $R$ with the induced ordering from $\subseteqq$, where we identify two such subsets $Y$ and $Z$ if their symmetric difference $Y \triangle Z$ is in $\mathcal{I}$. By a result of Taylor [2] we have $|\mathcal{P}(R)/\mathcal{I}| > 2^\omega$ because $\mathcal{I}$ is uniform $\sigma$-ideal. On the other hand

$$|\mathcal{P}(R)/\mathcal{I}| = |\mathcal{B} \triangle \mathcal{I}/\mathcal{I}| \leqq |\mathcal{B}| \leqq 2^\omega.$$

Hence a contradiction.

The above reasoning works also for Theorem B (ii) because Taylor [2] proved that $2^\lambda \leqq 2^\omega$ for all $\lambda < 2^\omega$ implies $|\mathcal{P}(R)/\mathcal{I}| = 2^{2^\omega}$.

Observe that we can not assume in Theorem B that $\mathcal{F}$ is only proper instead of that $\mathcal{F}$ is uniform. Indeed. Assume $2^{\omega_1} = 2^\omega$ (here $\omega_1$ is the first uncountable cardinal). Let $R = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$ and $|X_2| = \omega_1$. Let $\mathcal{I}$ be the $\sigma$-ideal on $R$ generated by $\mathcal{P}(X_1)$ and $[X_2]^{\leq\omega}$. Let $\mathcal{B}$ be the $\sigma$-field on $R$ generated by $\mathcal{P}(X_2) \cup \{X_1\}$ and let $\mathcal{F}$ be the filter dual to $\mathcal{I}$. We have $|\mathcal{B}| = 2^\omega$ and the $\sigma$-field generated by $\mathcal{B}$ and $\mathcal{F}$ is equal to $\mathcal{P}(R)$.

**Remark 2.** If there is a uniform $\sigma$-ideal $\mathscr{I}$ on $R$ such that $|\mathscr{P}(R)/\mathscr{I}| < 2^{2^\omega}$ then there is a $\sigma$-field $\mathscr{B}$ on $R$ such that $|\mathscr{B}| = |\mathscr{P}(R)/\mathscr{I}| < 2^{2^\omega}$ and $\mathscr{B} \vartriangle \mathscr{I} = \mathscr{P}(R)$.

Indeed. Let $\lambda = |\mathscr{P}(R)/\mathscr{I}|$. Let $\langle X_t \colon t < \lambda \rangle$ be a selector from the family $\mathscr{P}(R)/\mathscr{I}$. By a result of Comfort-Hager and Monk-Sparks (compare [2] and references there) we have $\lambda^\omega = \lambda$ because $\mathscr{P}(R)/\mathscr{I}$ is an infinite $\sigma$-complete Boolean algebra. Hence the $\sigma$-field $\mathscr{B}$ generated by the family $\langle X_t \colon t < \lambda \rangle$ has cardinality $\lambda \cdot \omega_1 = \lambda$. It is evident that $\mathscr{B} \vartriangle \mathscr{I} = \mathscr{P}(R)$.

## References


[1] GRZEGOREK E. and WĘGLORZ B., Extensions of filters and fields of sets I, J. Austral. Math. Soc. 25 (Series A), (1978), 275−290.

[2] TAYLOR A. D., The cardinality of reduced power set algebras, Proc. Amer. Math. Soc. 103 (1) (1988), 277−280.

[3] ULAM S. M., A collection of mathematical problems, Interscience Publishers, Inc. New York 1960.

[4] −, The scottish Book, A LASL Monograph 1977.

[5] −, Combinatorial analysis in infinite sets and some physical theories, SIAM Review, Vol. 6, No 4, 1964, 343−355.

[6] WĘGLORZ B., Extensions of filters to Ulam filters, Bull. Ac. Pol.: Math. 27 (1) (1979), 11−14.