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Characterization of Baire-One Functions Between Topological Spaces

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Let X be a normal topological space and Y be a metric space. We give several sufficient conditions under which the functions of the first Baire class from X into Y are characterized by their F_σ-measurability and strong σ-discreteness. For example, this happens if Y is arcwise connected and locally arcwise connected, or if Y contains a dense subspace Y_1 such that all open balls in Y_1 are arcwise connected. Other sufficient conditions are stated in terms of extendability of continuous functions from zero-subsets of X into Y to the whole X.

Introduction

Let $\mathscr{C}(X, Y)$ be the set of all continuous functions from a topological space X into a topological space Y. We use the following notation: $\mathscr{B}_1(X, Y) = \{f : X \to Y;$ f is a pointwise limit of a sequence from $\mathscr{C}(X, Y)\}$ and $\mathscr{F}_\sigma(X, Y) = \{f : X \to Y;$ $f^{-1}(G)$ is an $\mathscr{F}_\sigma$ set for any open $G \subset Y\}$. The elements of $\mathscr{B}_1(X, Y)$ are called *functions of the first Baire class*, and those of $\mathscr{F}_\sigma(X, Y)$ are called *functions of the first Borel class* or *$\mathscr{F}_\sigma$-measurable functions.*

It is easy to prove that $\mathscr{B}_1(X, Y) \subset \mathscr{F}_\sigma(X, Y)$ for any topological space X and any metric space Y (cf. Proposition 1.10), but the two classes do not coincide in general: the characteristic function of any nonempty proper closed subset of $[0, 1]$ belongs to $\mathscr{F}_\sigma([0, 1], \{0, 1\}) \setminus \mathscr{B}_1([0, 1], \{0,1\})$ (note that $\mathscr{B}_1([0, 1], \{0, 1\})$ contains constant functions only).

The research of relations between Baire-one functions and $\mathscr{F}_\sigma$-measurable functions begins with Baire's paper [1] from 1899, which contains results of his PhD. thesis.

The equality $\mathscr{B}_1(X, Y) = \mathscr{F}_\sigma(X, Y)$ holds in any of the following situations: (I) X is an interval of reals $\mathbb{R}$, $Y = \mathbb{R}$ (Baire [1]); (II) X is metric, $Y = \mathbb{R}$ (Lebesgue [10]); (III) X is metric, $Y = [0, 1]^n$ ($n \in \mathbb{N}$) or $Y = [0, 1]^{\mathbb{N}}$ ([7, p. 391]); (IV) X is

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metric, Y is a separable convex subset of a Banach space (Rolewicz [13]); (V) X is a complete metric space, Y is a Banach space (Stegall [14]); (VI) X is a normal topological space, $Y = \mathbb{R}$ (Laczkovich [9] without proof, for a proof see [11, Exercise 3.A.1]; the result was proved independently in [5]).

All these results, except (V), deal with Y separable. In fact, the functions from $\mathscr{B}_1(X, Y)$ are of countable character in some sense: they are limits of sequences of continuous functions. R. Hansell [4, § 3] introduced the notion of a σ-discrete function and observed that the functions from $\mathscr{B}_1(X, Y)$ are always σ-discrete.

A family of subsets of a topological space is called *discrete* if each point of the space has a neighborhood that meets at most one of the sets of the family. A family of sets is said to be *σ-discrete* if the family is the union of countably many discrete families. A family of sets in X is a *base* for $f : X \to Y$ if $f^{-1}(G)$ is a union of sets from the family whenever G is an open subset of Y. A function is said to be *σ-discrete* if it has a σ-discrete base. We shall denote by $\Sigma(X, Y)$ the set of all σ-discrete functions from X to Y.

Hansell [4] proved that Borel measurable functions defined on a complete metric space and functions with separable ranges are σ-discrete. Hence $\mathscr{F}_\sigma(X, Y) = F_\sigma(X, Y) \cap \Sigma(X, Y)$ holds in all the situations (I)–(VI) above.

The equality $\mathscr{B}_1(X, Y) = \mathscr{F}_\sigma(X, Y) \cap \Sigma(X, Y)$ holds in any of the following situations:

(VII) X and Y are metric spaces, every continuous function from a closed subset of X into Y can be extended continuously to X, and for each $y \in Y$ and each neighborhood U of y there is a neighborhood V of y such that each continuous function from a closed $F \subset X$ into V admits an extension from $\mathscr{C}(X, U)$ (Rogers [12]); (VIII) X is a paracompact space in which open sets are $\mathscr{F}_\sigma$, Y is a Banach space (Jayne, Orihuela, Pallarés and Vera [6]); (IX) X is collectionwise normal (i.e., for each discrete family $\{F_\alpha; \alpha \in \mathfrak{A}\}$ of closed sets there is a discrete family $\{G_\alpha; \alpha \in \mathfrak{A}\}$ of open sets with $F_\alpha \subset G_\alpha$ for any $\alpha \in \mathfrak{A}$), Y is a closed convex subset of a Banach space (Hansell [5]); (X) X is metric, Y is a metric space which is arcwise connected and locally arcwise connected (Fosgerau [3]).

A complete metric space Y is locally arcwise connected (and arcwise connected) if and only if Y is locally connected (and connected) by [8, p. 254] (and [3, proof of Thm.2]). M. Fosgerau [3] also proved that this property of Y is not only sufficient but also necessary for the equality $\mathscr{B}_1([0, 1], Y) = \mathscr{F}_\sigma([0, 1], Y)$. Namely, he proved the following theorem.

Theorem F. Let Y be a complete metric space and let X_0 be a metric space containing a homeomorphic copy of $[0, 1]$. Then the following assertions are equivalent:

(a) Y is connected and locally connected;
(b) $\mathscr{B}_1([0, 1], Y) = \mathscr{F}_\sigma([0, 1], Y)$;
(c) $\mathscr{B}_1(X_0, Y) = \mathscr{F}_\sigma(X_0, Y) \cap \Sigma(X_0, Y)$;
(d) $\mathscr{B}_1(X, Y) = \mathscr{F}_\sigma(X, Y) \cap \Sigma(X, Y)$ for all metric spaces X.

The aim of the present paper is to extend the above mentioned results (VII) and (X) (and hence all the result (I)–(X)) to the case when X is a normal topological space. Two problems arise. The proof requires to consider differences of closed sets, and such differences are not necessarily $\mathscr{F}_\sigma$ in non-metric spaces. The second problem concerns σ-discrete functions. In a metric space X each σ-discrete cover of X by $\mathscr{F}_\sigma$ sets has a refinement which covers X and is the union of countably many uniformly discrete families of $\mathscr{F}_\sigma$ sets. (A family of sets is uniformly discrete if there is a positive number less than the distance of any two distinct sets of the family.) This cannot be done in non-metric spaces (and this is the reason why (IX) requires X to be collectionwise normal, and (VIII) paracompact (and hence collectionwise normal, too [2, p. 214])).

The idea how to avoid the first obstacle is contained in [11]: instead of differences of closed sets it is possible to consider differences of zero-sets (i.e., sets of the form $\varphi^{-1}(0)$ where φ is a continuous real function). Such differences are $\mathscr{F}_\sigma$, even countable unions of zero-sets. The key is provided by Proposition 1.8.

As for the second problem, we observed that the functions of $\mathscr{B}_1(X, Y)$ are not only σ-discrete but „strongly σ-discrete". This notion (see Definition 1.2) coincides with σ-discreteness in collectionwise normal (and hence also in paracompact and in metric) spaces.

The proofs have much in common: they require to extend continuous functions. Not all, but only some of them. From this reason we define a (rather technical) property $(\mathscr{E})$ for couples of spaces (X, Y). We prove that $(\mathscr{E})$ is sufficient for $\mathscr{B}_1(X, Y) = \mathscr{F}_\sigma(X, Y) \cap \Sigma^*(X, Y)$ where $\Sigma^*(X, Y)$ denotes the class of strongly σ-discrete functions, and we show that if X is normal and Y is like in (VII) or in (X) then (X, Y) has the property $(\mathscr{E})$. We state other sufficient conditions for $(\mathscr{E})$ in terms of properties of a dense subspace of Y. These conditions are new and they cover some cases which were not covered by the results (I)–(X).

The main results of the present paper are contained in Theorem 3.2 and Theorem 3.7.

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1. Definitions and basic facts

1.1 Definition. A family $\mathscr{M}$ of subsets of a topological space X is called *strongly discrete* if there is a discrete (indexed) family $\{G_M; M \in \mathscr{M}\}$ of open sets such that $\overline{M} \subset G_M$ for any $M \in \mathscr{M}$. A family $\mathscr{M}$ is said to be *strongly σ-discrete* if $\mathscr{M}$ can be decomposed into countably many strongly discrete subfamilies. $\mathscr{M}$ is called *strongly discretely σ-decomposable* (shortly: *sdσd*) if each $M \in \mathscr{M}$ can be written in the form $M = \bigcup_{n=1}^{\infty} H_{M,n}$ and for each fixed $n \in \mathbb{N}$ the family $\{H_{M,n}; M \in \mathcal{M}\}$ is strongly discrete.

1.2 Definition. A function $f: X \to Y$ is said to be *strongly σ-discrete* if it has a strongly σ-discrete base. The class of all strongly σ-discrete functions from X into Y will be denoted by $\Sigma^*(X, Y)$.

1.3 Remark. (i) Each strongly discrete family is discrete, hence $\Sigma^*(X, Y) \subset \subset \Sigma(X, Y)$.
(ii) In view of Definition 1.1, a space is collectionwise normal iff every discrete family of its subsets is strongly discrete. Therefore $\Sigma^*(X, Y) = \Sigma(X, Y)$ if X is collectionwise normal (in particular, if X is paracompact or metric).
(iii) Every strongly σ-discrete family is *sdσd*.
(iv) Each metric space has a σ-discrete base of open sets (cf. [7, p. 235]).

1.4 Definition. A subset A of a topological space X is a *zero-set* if $A = \varphi^{-1}(0)$ for some $\varphi \in \mathscr{C}(X, \mathbb{R})$. A is a *cozero-set* if its complement is a zero-set. We denote by $\mathscr{Z}$, Coz, $\mathscr{Z}_\sigma$, Coz_δ respectively the families of all zero-sets, cozero-sets, countable unions of zero-sets, and countable intersections of cozero-sets.

1.5 Remark. (i) Zero-sets are closed. In metric spaces every closed set is a zero-set (consider φ equal to the distance from the set).
(ii) The class $\mathscr{Z}$ is closed under finite unions and finite intersections.
(iii) If F is a closed set in a metric space Y and $\varphi \in \mathscr{C}(X, Y)$, then $\varphi^{-1}(F) \in \mathscr{Z}$.
(iv) $Coz \subset \mathscr{Z}_\sigma$.

1.6 Lemma. Union of a strongly discrete family of zero-sets in a normal space X is again a zero-set.

Proof. Let $\mathcal{M}$ be our family. By Definition 1.1 there is a discrete family $\{G_M; M \in \mathcal{M}\}$ of open sets such that $M \subset G_M$ for any $M \in \mathcal{M}$. For $M \in \mathcal{M}$, let $\varphi_M \in \mathscr{C}(X, \mathbb{R})$ be such that $M = \varphi_M^{-1}(0)$. By the normality of X, for every $M \in \mathcal{M}$ there is $\psi_M \in \mathscr{C}(X, [0, 1])$ such that $\psi_M(M) = \{0\}$ and $\psi_M(X \setminus G_M) = \{1\}$. Define $f_M(x) = \min\{|\varphi_M(x)| + \psi_M(x), 1\}$ and put

$$f(x) = \begin{cases} f_M(x) & \text{for } x \in G_M, M \in \mathcal{M}, \\ 1 & \text{for } x \notin \cup\{G_M; M \in \mathcal{M}\}. \end{cases}$$

Then $f \in \mathscr{C}(X, \mathbb{R})$ and $f^{-1}(0) = \cup\mathcal{M}$. □

1.7 Lemma. Let X be a normal space, $A \subset B \subset X$, $A \in \mathscr{F}_\sigma$ and $B \in \mathscr{G}_\delta$. Then there exists $H \in Coz_\delta$ such that $A \subset H \subset B$.

Proof. First, let us prove the lemma for B open. We can write $A = \bigcup_{n=1}^{\infty} A_n$ with A_n closed for all n. The sets $X \setminus B$ and A_n are disjoint and closed, therefore there exists

$\varphi_n \in \mathscr{C}(X, [0, 1])$ with $\varphi_n(X \setminus B) = \{0\}$ and $\varphi_n(A_n) = \{1\}$. Put $\varphi = \sum_{n=1}^{\infty} 2^{-n}\varphi_n$ and $H = X \setminus \varphi^{-1}(0)$. Then H is a cozero set and $A \subset H \subset B$. The assertion for a general $\mathscr{G}_\delta$ set B follows easily from the particular case proved above. □

1.8 Proposition. Let X be a normal space, Y be metric, and $f \in \mathscr{F}_\sigma(X, Y)$. Then $f^{-1}(G) \in \mathscr{Z}_\sigma$ for any open $G \subset Y$.

Proof. Let $F \subset Y$ be closed, $d(y) = \operatorname{dist}(y, F)$ for any $y \in Y$. Then the function $g = d \circ f$ is in $\mathscr{F}_\sigma(X, \mathbb{R})$ and we have

$$f^{-1}(F) = g^{-1}(0) = \bigcap_{n=1}^{\infty} g^{-1}((-1/n, 1/n)) = \bigcap_{n=1}^{\infty} g^{-1}([-1/n, 1/n]) .$$

The sets $A_n = g^{-1}((-1/n, 1/n))$ are $\mathscr{F}_\sigma$, the sets $B_n = g^{-1}([-1/n, 1/n])$ are $\mathscr{G}_\delta$, and $A_n \subset B_n$ for all n. By Lemma 1.7 there exist sets $H_n \in Coz_\delta$ such that $A_n \subset H_n \subset B_n$. Consequently, $f^{-1}(F) = \cup H_n \in Coz_\delta$ for any closed set F. Passing to complements completes the proof. □

1.9 Lemma. Let $\mathscr{U}$ be a σ-discrete family of open sets in a space Y, and $f \in \Sigma^*(X, Y)$. Then the family $\{f^{-1}(U); U \in \mathscr{U}\}$ is *sdσd*.

Proof. Let $\mathscr{B} = \bigcup_{m=1}^{\infty} \mathscr{B}_m$ be a base for f such that each $\mathscr{B}_m$ is strongly discrete. Write $\mathscr{U} = \bigcup_{n=1}^{\infty} \mathscr{U}_n$ where each $\mathscr{U}_n$ is discrete. For $U \in \mathscr{U}$, $m, n \in \mathbb{N}$ put

$$H_{U,m,n} = \begin{cases} \cup\{B \in \mathscr{B}_m; B \subset f^{-1}(U)\} & \text{for} \quad U \in \mathscr{U}_n , \\ \emptyset & \text{for} \quad U \notin \mathscr{U}_n . \end{cases}$$

Obviously, $f^{-1}(U) = \bigcup_{m,n\in\mathbb{N}} H_{U,m,n}$. Moreover, for fixed m, n, the family $\{H_{U,m,n}; U \in \mathscr{U}\}$ is strongly discrete since $\mathscr{B}_m$ is strongly discrete and $\{f^{-1}(U); U \in \mathscr{U}_n\}$ is disjoint. □

1.10 Proposition. Let X be a topological space and let Y be a metric space. Then $\mathscr{B}_1(X, Y) \subset \mathscr{F}_\sigma(X, Y) \cap \Sigma^*(X, Y)$.

Proof. Let $f(x) = \lim f_m(x)$ for all $x \in X$, $f_m \in \mathscr{C}(X, Y)$ for all m. Let $U \subset Y$ be open. There exists closed sets U_k $(k \in \mathbb{N})$ such that $U = \bigcup_{k=1}^{\infty} U_k$, and $U_k \subset \operatorname{int}(U_{k+1})$ for all k. Then

$$f^{-1}(U) = \bigcup_{k,m\in\mathbb{N}} \bigcap_{j\geq m} f_j^{-1}(U_k) .$$

The sets $F^U_{k,m} = \bigcap_{j\geq m} f_j^{-1}(U_k)$ are closed. Consequently $f \in \mathscr{F}_\sigma(X, Y)$.

Let $\mathscr{U} = \bigcup_{n=1}^{\infty} \mathscr{U}_n$ be an open base for the topology of Y with $\mathscr{U}_n$ discrete for all n (cf. Remark 1.3(iv)). For fixed $n, k, m \in \mathbb{N}$ the family $\mathscr{B}_{n,k,m} = \{F^U_{k,m}; U \in \mathscr{U}_n\}$ is strongly discrete since $F^U_{k,m} \subset f_m^{-1}(U)$ and $\{f_m^{-1}(U); U \in \mathscr{U}_n\}$ is a discrete family of open sets. It is easy to see that $\mathscr{B} = \bigcup_{n,k,m} \mathscr{B}_{n,k,m}$ is a base for f. Thus $f \in \Sigma^*(X, Y)$. □

2. Strongly discretely σ-decomposable families of $\mathscr{Z}_\sigma$ sets

2.1 Lemma. Let $\mathscr{M}$ be a *sdσd* family of $\mathscr{Z}_\sigma$ subsets of a normal space X. Then the sets $H_{M,n}$ from Definition 1.1 can be chosen so that they are zero-sets.

Proof. By Definition 1.1, there exist sets $H_{M,n}$ and open sets $G_{M,n}$ such that $\overline{H}_{M,n} \subset G_{M,n}$ for any $M \in \mathscr{M}$, $n \in \mathbb{N}$, $M = \bigcup_{n=1}^{\infty} H_{M,n}$ for any $M \in \mathscr{M}$, and the family $\{G_{M,n}; M \in \mathscr{M}\}$ is discrete for any $n \in \mathbb{N}$. Since X is normal it is easy to find zero-sets $Z_{M,n}$ with $\overline{H}_{M,n} \subset Z_{M,n} \subset G_{M,n}$ for $M \in \mathscr{M}$, $n \in \mathbb{N}$. Each $M \in \mathscr{M}$ is in $\mathscr{Z}_\sigma$, therefore it can be written in the form $M = \bigcup_{k=1}^{\infty} F_{M,k}$ with $F_{M,k} \in \mathscr{Z}$ for all k. So we have

$$M = \bigcup_n H_{M,n} = \bigcup_n \bigcup_k (H_{M,n} \cap F_{M,k}) \subset \bigcup_n \bigcup_k (Z_{M,n} \cap F_{M,k}) \subset \bigcup_k F_{M,k} = M\,,$$

hence $M = \bigcup_n \bigcup_k (Z_{M,n} \cap F_{M,k})$. The sets in the last union are zero-sets and for fixed n, k the family $\{Z_{M,n} \cap F_{M,k}; M \in \mathscr{M}\}$ is strongly discrete since $Z_{M,n} \cap F_{M,k} \subset \subset G_{M,n}$. $\square$

2.2 Proposition (Reduction lemmai. Let $\{M_\alpha; \alpha \in \mathfrak{A}\}$ be a *sdσd* family of $\mathscr{Z}_\sigma$ sets in a normal space X. Then there exists a disjoint *sdσd* family $\{F_\alpha; \alpha \in \mathfrak{A}\}$ of $\mathscr{Z}_\sigma$ sets such that

(a) $F_\alpha \subset M_\alpha$ for all $\alpha \in \mathfrak{A}$, and

(b) $\bigcup_{\alpha \in \mathfrak{A}} F_\alpha = \bigcup_{\alpha \in \mathfrak{A}} M_\alpha$.

Prool. By Lemma 2.1, for any $\alpha \in \mathfrak{A}$ we can write $M_\alpha = \bigcup_{n=1}^{\infty} M_{\alpha,n}$ where $M_{\alpha,n}$ are zero-sets, and for each fixed n the family $\{M_{\alpha,n}; \alpha \in \mathfrak{A}\}$ is strongly discrete. Define by induction

$$F_{\alpha,1} = M_{\alpha,1} \quad \text{for all} \quad \alpha \in \mathfrak{A}\,,$$

$$F_{\alpha,n+1} = M_{\alpha,n+1} \setminus \bigcup_{k=1}^{n} \bigcup_{\alpha \in \mathfrak{A}} M_{\alpha,k} \quad \text{for all} \quad \alpha \in \mathfrak{A}\,.$$

By Lemma 1.6 and Remark 1.5(ii) the last union is a zero-set. Hence, by Remark 1.5(iv), $F_{\alpha,n} \in \mathscr{Z}_\sigma$ for any $\alpha \in \mathfrak{A}$, $n \in \mathbb{N}$. It is clear that $\{F'_{\alpha,n}; \alpha \in \mathfrak{A}, n \in \mathbb{N}\}$ is a disjoint cover of $\bigcup_\alpha M_\alpha$, and $F_{\alpha,n} \subset M_{\alpha,n}$ for all α, n. Consequently the sets $F_\alpha = \bigcup_{n=1}^{\infty} F_{\alpha,n}$ $(\alpha \in \mathfrak{A})$ have the required properties. $\square$

2.3 Lemma. Let X be a normal space, and for any $s \in \mathbb{N}$, let $\mathscr{M}^s \subset \mathscr{Z}_\sigma$ be a disjoint *sdσd* family that covers X. Then there exist families $\mathscr{A}_n^s (s, n \in \mathbb{N})$ satisfying the following properties:

(a) $\mathscr{A}_n^s$ is a strongly discrete family of zero-sets;

(b) for each $F \in \mathscr{A}_n^s$ there exists (necessarily unique) $M \in \mathscr{M}^s$ with $F \subset M$;

(c) $\bigcup_{n=1}^{\infty} (\cup \mathscr{A}_n^s) = X$;

(d) $\cup \mathscr{A}_n^s \subset \cup \mathscr{A}_{n+1}^s$;

(e) for each $H \in \mathscr{A}_n^{s+1}$ there exists (necessarily unique) $F \in \mathscr{A}_n^s$ with $H \subset F$.

Proof. 1. Fix $s \in \mathbb{N}$. By Lemma 2.1, each $M \in \mathscr{M}^s$ can be written in the form $M = \bigcup_{n=1}^{\infty} Z_{M,n}$ where $Z_{M,n} \in \mathscr{Z}$ for each n, so that $\{Z_{M,n}; M \in \mathscr{M}^s\}$ is strongly discrete for any fixed n. Denote $Z_n = \cup \{Z_{M,n}; M \in \mathscr{M}^s\}$.

Let $m \in \mathbb{N}$, $m \geqq 2$, $j \in \mathbb{N}$. By Lemma 1.6 and Remark 1.5(ii) there exists $\varphi_m \in \mathscr{C}(X, [0, +\infty))$ such that $\varphi_m^{-1}(0) = \bigcup_{i=1}^{m-1} Z_i$. Put

$$H_{m,j} = \bigcup_{M \in \mathscr{M}} (Z_{M,n} \cap \varphi_m^{-1}([1/j, +\infty))),$$

and observe that $H_{m,j}$ is the union of a strongly disctete family of zero-sets. Moreover

$$H_{m,j} \subset Z_m \setminus \bigcup_{i=1}^{m-1} Z_i . \tag{1}$$

Put $B_1^s = Z_1$, $B_n^s = Z_1 \cup \bigcup_{k=2}^{n} H_{k,n}$ for $n \geqq 2$. Then B_n^s is a disjoint union of finitely many sets, each of which is the union of a strongly discrete family of zero-sets. The normality of X implies that B_n^s is the union of a strongly discrete family $\mathscr{B}_n^s$ of zero-sets, where

$$\mathscr{B}_n^s = \{Z_{M,1}; M \in \mathscr{M}^s\} \cup \bigcup_{k=2}^{n} \{Z_{M,k} \cap \varphi_k^{-1}([1/n, +\infty)); M \in \mathscr{M}^s\} .$$

For $n \geqq 2$ we have

$$B_1^s \subset B_n^s = Z_1 \cup \bigcup_{k=2}^{n} H_{k,n} \subset Z_1 \cup \bigcup_{k=2}^{n} H_{k,n+1} \subset B_{n+1}^s . \tag{2}$$

Moreover

$$X \setminus Z_1 = \bigcup_{n=2}^{\infty} \bigcup_{k=2}^{n} H_{k,n} . \tag{3}$$

(In fact, if $x \in X \setminus Z_1$ then there exists $k \geqq 2$ such that $x \in Z_k \setminus \bigcup_{i=1}^{k-1} Z_i = \bigcup_{M \in \mathscr{M}} (Z_{M,k} \cap \varphi_k^{-1}((0, +\infty))) = \bigcup_{j=1}^{\infty} H_{k,j}$. So $x \in H_{k,j}$ for some j. Take $n \geqq \max\{k, j\}$ and observe that $H_{k,j} \subset H_{k,n}$. We have found $k \geqq 2$ and $n \geqq k$ such that $x \in H_{k,n}$.)

Using (3) we get

$$\bigcup_{n=1}^{\infty} B_n^s = Z_1 \cup \bigcup_{n=2}^{\infty} \bigcup_{k=2}^{n} H_{k,n} = X . \tag{4}$$

2. Define $\tilde{\mathscr{A}}_n^s = \{F_1 \cap \ldots \cap F_s; F_i \in \mathscr{B}_n^i, 1 \leqq i \leqq s\}$, $A_n^s = \cup \tilde{\mathscr{A}}_n^s$. Hence $A_n^s = \bigcap_{i=1}^{s} B_n^i$. By (2), $A_n^s \subset A_{n+1}^s$ for all $s, n \in \mathbb{N}$. Moreover $\bigcup_{n=1}^{\infty} A_n^s = X$ for all $s \in \mathbb{N}$.

(In fact, if $x \in X$ then for $1 \leqq i \leqq s$ there is $n_i \in \mathbb{N}$ with $x \in B^i_{n_i}$. For $n = \max . \{n_i; 1 \leqq i \leqq s\}$ we have $x \in \bigcap_{i=1}^{s} B^i_n = A^s_n$.) Obviously $A^{s+1}_n \subset A^s_n$ for all $s, n \in \mathbb{N}$.

Each set A^s_n is the union of the strongly discrete family $\tilde{\mathscr{A}}^s_n$ of zero-sets. For fixed n we can inductively define strongly discrete families $\mathscr{A}^s_n \subset \mathscr{Z}$ $(s \in \mathbb{N})$ such that each element of $\mathscr{A}^{s+1}_n$ is contained in some element of $\mathscr{A}^s_n$. It suffices to take

$$\mathscr{A}^1_n = \tilde{\mathscr{A}}^1_n\,, \quad \mathscr{A}^{s+1}_n = \{T_1 \cap T_2; T_1 \in \mathscr{A}^s_n, T_2 \in \tilde{\mathscr{A}}^{s+1}_n\}\,.$$

It is easy to see that the properties of the sets A^s_n imply (a), (c), (d), (e). It remains to show (b). If $F \in A^s_n$ then F is contained in some $T \in \tilde{\mathscr{A}}^s_n$, T is contained in some $B \in \mathscr{B}^s_n$, and finally, B is contained in some $M \in \mathscr{M}^s$. The proof is complete. □

By a refinement of a family $\mathscr{A}$ of sets we mean any family $\mathscr{B}$ of sets such that $\cup\mathscr{B} = \cup\mathscr{A}$ and any element of $\mathscr{B}$ is contained in some element of $\mathscr{A}$.

2.4 Remark. Let each of $\mathscr{M}$, $\mathscr{N}$ be a disjoint *sdσd* family of $\mathscr{Z}_\sigma$ sets that covers X. Then the family $\{M \cap N; M \in \mathscr{M}, N \in \mathscr{N}\}$ has the same properties. (Indeed, by Lemma 2.1 there exist zero-sets $F_{M,i}$ and $H_{N,j}$ and open sets $U_{M,i}$ and $V_{N,j}$ such that $F_{M,i} \subset U_{M,i}$ and $H_{N,j} \subset V_{N,j}$ for $M \in \mathscr{M}$, $N \in \mathscr{N}$, $i, j \in \mathbb{N}$, $M = \bigcup_i F_{M,i}$, $N = \bigcup_j H_{N,j}$, and for fixed i, j the families $\{U_{M,i}; M \in \mathscr{M}\}$, $\{V_{N,j}; N \in \mathscr{N}\}$ are discrete. It is easy to see that $M \cap N = \bigcup_{i,j}(F_{M,i} \cap H_{N,j})$ and the family $\{U_{M,i} \cap V_{N,j}; M \in \mathscr{M}, N \in \mathscr{N}\}$ is discrete.)

2.5 Theorem. Let X be normal, Y be metric, and $f \in \mathscr{F}_\sigma(X, Y) \cap \Sigma^*(X, Y)$. For each $s \in \mathbb{N}$, let $\mathscr{U}^s$ be an open cover of Y. Then there exist families $\mathscr{M}^s$ of sets in X $(s \in \mathbb{N})$ and open sets $W(x, s)$ $(x \in X, s \in \mathbb{N})$ with the following properties:
(a) $\mathscr{M}^s \subset \mathscr{Z}_\sigma$ and $\mathscr{M}^s$ is a disjoint *sdσd* cover of X;
(b) $W(x_1, s) = W(x_2, s)$ whenever $x_1, x_2 \in M \in \mathscr{M}^s$;
(c) $W(x, s)$ is contained in some element of $\mathscr{U}^s$;
(d) $W(x, s + 1) \subset W(x, s)$;
(e) $f(x) \in W(x, s)$.

Proof. Let us proceed by induction with respect to s. Let $\mathscr{W}^1 = \{W_\alpha; \alpha \in \mathfrak{A}_1\}$ be a σ-discrete open refinement of $\mathscr{U}^1$ (cf. Remark 1.3(iv)). By Proposition 1.8 and Lemma 1.9, $\{f^{-1}(W_\alpha); \alpha \in \mathfrak{A}_1\}$ consists of $\mathscr{Z}_\sigma$ sets and is *sdσd*. By Proposition 2.2 there exists a disjoint *sdσd* family $\mathscr{M}^1 = \{M_\alpha; \alpha \in \mathfrak{A}_1\} \subset \mathscr{Z}_\sigma$ such that $M_\alpha \subset f^{-1}(W_\alpha)$ for all $\alpha \in \mathfrak{A}_1$ and $\cup\mathscr{M}^1 = X$. We can define $W(x, 1) = W_\alpha$ for $x \in M_\alpha$, $\alpha \in \mathfrak{A}_1$. Clearly $f(x) \in W(x, 1)$.

Suppose that we have already defined disjoint index sets $\mathfrak{A}_i$, *sdσd* families $\mathscr{M}^i = \{M_\alpha; \alpha \in \mathfrak{A}_i\}$ of $\mathscr{Z}_\sigma$ sets, and open sets $W(x, i)$ $(x \in X)$ for $i = 1, 2, \ldots, s$, such that each $\mathscr{M}^i$ is disjoint and covers X, $W(., i)$ is constant on each member of $\mathscr{M}^i$, and for every $x \in X$ each $W(x, i)$ is contained in an element of $\mathscr{U}^i$, $f(x) \in W(x, s) \subset W(x, s - 1) \subset \ldots \subset W(x, 1)$.

Let $\mathscr{W}^{s+1} = \{W_\beta; \beta \in \mathfrak{B}_{s+1}\}$ be a σ-discrete open refinement of $\mathscr{U}^{s+1}$. Then, as above, $\{f^{-1}(W_\beta); \beta \in \mathfrak{B}_{s+1}\} \subset \mathscr{Z}_\sigma$ is *sdσd*. Hence by Proposition 2.2 there exists a disjoint family $\mathscr{N}^{s+1} = \{N_\beta; \beta \in \mathfrak{B}_{s+1}\} \subset \mathscr{Z}_\sigma$ such that $N_\beta \subset f^{-1}(W_\beta)$ for all $\beta \in \mathfrak{B}_{s+1}$ and $\cup \mathscr{N}^{s+1} = X$. We define $V(x, s+1) = W_\beta$ for $x \in N_\beta$, $\beta \in \mathfrak{B}_{s+1}$. Obviously $f(x) \in V(x, s+1)$ for all x. Define $\mathscr{M}^{s+1} = \{M_\alpha \cap N_\beta; \alpha \in \mathfrak{A}_s, \beta \in \mathfrak{B}_{s+1}\}$. By Remark 2.4, $\mathscr{M}^{s+1}$ is a disjoint *sdσd* family of $\mathscr{Z}_\sigma$ sets that covers X. For $x \in M_\alpha \cap \cap N_\beta$, $\alpha \in \mathfrak{A}_s$, $\beta \in \mathfrak{B}_{s+1}$ define $W(x, s+1) = W(x, s) \cap V(x, s+1)$. Then $W(., s+1)$ is constant on each $M_\alpha \cap N_\beta$, since $W(., s)$, $V(., s+1)$ are constant respectively on M_α, N_β. The other required properties are evident. The induction is complete (obviously we can write $\mathscr{M}^{s+1} = \{M_\alpha; \alpha \in \mathfrak{A}_{s+1}\}$ where $\mathfrak{A}_{s+1} = = A_s \times B_{s+1}$). □

3. The property ($\mathscr{E}$)

3.1 Definition. We shall say that a couple (X, Y) of spaces satisfies the *property* ($\mathscr{E}$) if X is normal, Y is metric, and for each zero-set $F \subset X$ there is a nonempty set $\Phi(F) \subset \mathscr{C}(X, Y)$ such that the following properties are satisfied:

(i) $\Phi(F_1) \subset \Phi(F_2)$ whenever $F_1 \supset F_2$;

(ii) there exists $f_0 \in \Phi(X)$ such that for every pair F_1, F_2 of disjoint zero-sets in X and every open $V \subset Y$ there exists $f \in \Phi(F_1)$ with $f(F_1) \subset V$ and $f|_{F_2} = f_0|_{F_2}$;

(iii) for any $y \in Y$ and any $\varepsilon > 0$ there exists a neighborhood U of y satisfying: if F_1, F_2 are two disjoint zero-sets in X, $f \in \Phi(F_1)$, $f(F_1) \subset U$ and V is an open subset of U, then there exists $g \in \Phi(F_1)$ with $g(F_1) \subset V$, $g|_{F_2} = f|_{F_2}$ and $d(f(x)), g(x)) < \varepsilon$ for all $x \in X$.

3.2 Theorem. If a couple (X, Y) satisfies the property ($\mathscr{E}$), then $\mathscr{B}_1(X, Y) = = \mathscr{F}_\sigma(X, Y) \cap \Sigma^*(X, Y)$.

Proof. One inclusion is contained in Proposition 1.10. To prove the other one, take an arbitrary function $f \in \mathscr{F}_\sigma(X, Y) \cap \Sigma^*(X, Y)$. Choose a sequence $\{\varepsilon_s\} \subset \subset (0, +\infty)$ so that $\sum_{s=1}^{\infty} \varepsilon_s < +\infty$. For any $y \in Y$ there exists aneighborhood $U = U_y^s$ satisfying the property (iii) from Definition 3.1 with $\varepsilon = \varepsilon_s$. Without any loss of generality we can suppose that U_y^s is open and diam $(U_y^s) < \varepsilon_s$.

Let $\mathscr{M}^s$ and $W(x, s)$ $(s \in \mathbb{N}, x \in X)$ be the families and the open sets produced by Lemma 2.5 for the open coverings $\mathscr{U}^s = \{U_y^s; y \in Y\}$. They have the following properties:

(+) $\mathscr{M}^s \subset \mathscr{Z}_\sigma$ is a disjoint *sdσd* cover of X;

(++) $W(., s)$ is constant on each element of $\mathscr{M}^s$;

(+++) diam $(W(x, s)) < \varepsilon_s$ and $W(x, s)$ contains $f(x)$;

(++++) if F_1, F_2 are two disjoint zero-sets, $f_1 \in \Phi(F_1)$, $f_1(F_1) \subset W(x, s)$, then there exists $g \in \Phi(F_1)$ such that $g(F_1) \subset W(x, s+1)$, $g|_{F_2} = f_1|_{F_2}$, $d(f_1(x), g(x)) < \varepsilon_s$ for all x.

Let $\mathscr{A}_n^s$ be the families from Lemma 2.3 applied to the families $\mathscr{M}^s$. Since each $\mathscr{A}_n^s$ is a strongly discrete family of zero-sets, there exist discrete families $\mathscr{D}_n^s = \{U_F; F \in \mathscr{A}_n^s\}$ of open sets such that $F \subset U_F$ whenever $F \in \mathscr{A}_n^s$, $s, n \in \mathbb{N}$. Because of the property (e) from Lemma 2.3, we can suppose that any element of $\mathscr{D}_n^{s+1}$ is contained in an element of $\mathscr{D}_n^s$. In other words, $U_H \subset U_F$ whenever $H \in \mathscr{A}_n^{s+1}$, $F \in \mathscr{A}_n^s$, $H \subset F$. Moreover, it is possible to suppose $\mathscr{D}_n^s \subset Coz$.

Fix $n \in \mathbb{N}$. We shall inductively construct functions $h_{s,n} \in \mathscr{C}(X, Y)$ $(s \in \mathbb{N})$ with the property:

(§) $h_{s,n}(F) \subset W(F, s)$, and $h_{s,n}$ coincides on U_F with a function $g_F \in \Phi(F)$ whenever $F \in \mathscr{A}_n^s$.

(In view of $(++)$ the meaning of $W(F, s)$ is clear.)

$s = 1$. Let $f_0 \in \Phi(X)$ be as in Definition 3.1(ii). For any $F \in \mathscr{A}_n^1$ there exists $g_F \in \Phi(F)$ such that $g_F(F) \subset W(F, 1)$ and $g_F|_{X \setminus U_F} = f_0|_{X \setminus U_F}$. So it is possible to define $h_{1,n} \in \mathscr{C}(X, Y)$ by the formula

$$h_{1,n}(x) = \begin{cases} g_F(x) & \text{if } x \in U_F,\ F \in \mathscr{A}_n^1\ ; \\ f_0(x) & \text{if } x \in X \setminus \cup \mathscr{D}_n^1\ . \end{cases}$$

Suppose we have already defined $h_{1,n}, h_{2,n}, \ldots, h_{s,n}$. For any $H \in \mathscr{A}_n^{s+1}$ there is (by Lemma 2.3(e)) a unique $F \in \mathscr{A}_n^s$ with $H \subset F$ (and also $U_H \subset H_F$). Let g_F be as in (§). Then $g_F \in \Phi(H)$ and $g_F(H) \subset W(F, s)$. By $(++++)$ there exists a function $g_H \in \Phi(H)$ with $g_H(H) \subset W(H, s+1)$, $g_H|_{X \setminus U_H} = g_F|_{X \setminus U_H}$ and $d(g_H(x), g_F(x)) < \varepsilon_s$. Define $h_{s+1,n} \in \mathscr{C}(X, Y)$ by

$$h_{s+1,n}(x) = \begin{cases} g_H(x) & \text{if } x \in U_H,\ H \in \mathscr{A}_n^{s+1}\ ; \\ h_{s,n}(x) & \text{if } x \in X \setminus \cup \mathscr{D}_n^{s+1}\ . \end{cases}$$

The induction is done.

The functions $h_{s,n}$ satisfy $d(h_{s,n}(x), h_{s+1,n}(x)) < \varepsilon_s$ for $x \in X$, and $h_{s,n}(x) \in W(x, s)$ for $x \in \cup \mathscr{A}_n^s$. We shall show that the diagonal sequence $\{h_{n,n}\}$ converges pointwise to f.

Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Choose $s \in \mathbb{N}$ so that $\sum_{i=s}^{\infty} \varepsilon_s < \varepsilon$. By the properties (c), (d) from Lemma 2.3, there exists an index $n_0 > s$ such that $x \in \cup \mathscr{A}_n^s$ for all $n \geqq n_0$. For $n \geqq n_0$ we have

$$d(h_{n,n}(x), f(x)) \leqq d(h_{s,n}(x), f(x)) + \sum_{i=s}^{n-1} d(h_{i,n}(x), h_{i+1,n}(x)) \leqq$$
$$\leqq \operatorname{diam}(W(x, s)) + \sum_{i=s}^{n-1} \varepsilon_i < \varepsilon_s + \sum_{i=s}^{\infty} \varepsilon_i < 2\varepsilon\ .$$

Consequently, $f \in \mathscr{B}_1(X, Y)$. □

The following two theorems give sufficient conditions for the property $(\mathscr{E})$.

3.3 Theorem. Let X be normal, and let Y be a metric space containing a dense arcwise connected subset Y_1. Suppose that Y satisfies the following condition.

(A) There exists $D \subset Y$ with $D \cap Y_1$ dense in Y and such that for any $\varepsilon > 0$ and any $y \in Y$ there exists a neighborhood U of y satisfying: any two points of $D \cap U$ can be joined (in Y) with an arc of diameter less than ε.

Then (X, Y) satisfies the property $(\mathscr{E})$.

Proof. Choose $y_0 \in Y_1 \cap D$ and set $f_0(x) = y_0$ for all $x \in X$. For any zero-set $F \subset X$ define

$\Phi(F) = \{f \in \mathscr{C}(X, Y);$ there exist an open set $G \supset F$, $\varphi \in \mathscr{C}(G, [0, 1])$, $p \in \mathscr{C}([0, 1], Y)$ such that $f|G = p \circ \varphi$, $\varphi(F) = \{1\}$ and $p(1) \in D\}$.

Observe that any function from $\Phi(F)$ is constant on F, $f_0 \in \Phi(X)$, and $\Phi(F_1) \subset \Phi(F_2)$ whenever $F_1 \subset F_2$.

Let F_1, F_2 be two disjoint zero-sets in X and $V \subset Y$ be open. Choose an arbitrary $y_1 \in V \cap Y_1 \cap D$ and find $p \in \mathscr{C}([0, 1], Y)$ with $p(0) = y_0$ and $p(1) = y_1$. The space X is normal, so there exists $\varphi \in \mathscr{C}(X, [0, 1])$ with $\varphi(F_1) = \{1\}$ and $\varphi(F_2) = \{0\}$. Then the function $f = p \circ \varphi$ belongs to $\Phi(F_1)$ and satisfies $f(F_1) = \{y_1\} \subset V$ and $f|_{F_2} \equiv p(0) = y_0 \equiv f_0|_{F_2}$. Thus the condition (ii) from Definition 3.1 is verified.

Let us prove the condition (iii) of Definition 3.1. Let $y \in Y$ and $\varepsilon > 0$ be given. Let U be the neighborhood of y from (A). Suppose that F_1, F_2 are two disjoint zero-sets in X, $f \in \Phi(F_1)$, $f(F_1) \subset U$, V is an open subset of U. Take an open set $G \supset F_1$ and functions $\varphi \in \mathscr{C}(G, [0, 1])$, $p \in \mathscr{C}([0, 1], Y)$ such that $f = p \circ \varphi$ on G, $\varphi(F_1) = \{1\}$ and $p(1) = y_1 \in D$. We can suppose $\bar{G} \cap F_2 = \emptyset$. Choose arbitrarily $u \in V \cap D$ and find $q \in \mathscr{C}([0, 1], Y)$ with $q(0) = y_1$, $q(1) = u$ and $\operatorname{diam}(q([0, 1])) < \varepsilon$. The normality of X assures the existence of $\psi \in \mathscr{C}(X, [0, 1])$ with $\psi(F_1) = \{1\}$ and $\psi(X \setminus G) = \{0\}$. Let $\delta > 0$ be such that $d(p(s), p(t) < \varepsilon - \operatorname{diam}(q([0, 1]))$ whenever $|t - s| \leqq \delta$, $t, s \in [0, 1]$. Define $Q \in \mathscr{C}([0, 1 + \delta], Y)$ by $Q(t) = p(t)$ and $Q(1 + \delta t) = q(t)$ for $t \in [0, 1]$. The function

$$g(x) = \begin{cases} Q(\varphi(x) + \delta\psi(x)) & \text{for } x \in \bar{G}, \\ f(x) & \text{for } x \in X \setminus \bar{G} \end{cases}$$

is continuous, since for $x \in \partial G$ we have $g(x) = Q(\varphi(x)) = p(\varphi(x)) = f(x)$. Moreover, $g|F_2 = f|F_2$ since $F_2 \subset X \setminus \bar{G}$. Consequently, $g \in \Phi(F_1)$. It remains to show that $d(f(x), g(x)) < \varepsilon$ for all $x \in X$.

For $x \in X \setminus G$, $d(f(x), g(x)) = 0$. For $x \in G$ there are two possibilities.

α) $\varphi(x) + \delta\psi(x) > 1$. In this case $1 - \varphi(x) \leqq \delta$ and hence

$$d(f(x), g(x)) \leqq d(f(x), y_1) + d(y_1, g(x)) = d(p(\varphi(x)), p(1)) +$$

$$+ d\left(q(0), q\left(\frac{\varphi(x) + \delta\psi(x) - 1}{\delta}\right)\right) < [\varepsilon - \operatorname{diam}(q([0,1]))] + \operatorname{diam}(q([0,1])) = \varepsilon.$$

β) $\varphi(x) + \delta\psi(x) \leqq 1$. In this case $d(f(x), g(x)) = d(p(\varphi(x)), p(\varphi(x) + \delta\psi(x))) < \varepsilon - \operatorname{diam}(q([0, 1])) < \varepsilon$. □

3.4 Theorem. Let X be normal, and let Y be a metric space containing a dense subset Y_1 such that for any $y_1, y_2 \in Y_1$, each continuous function from a zero-set (in X) into $\{y_1, y_2\}$ admits an extension from $\mathscr{C}(X, Y)$. Suppose that the following condition is satisfied.

(A) There exists $D \subset Y$ with $D \cap Y_1$ dense in Y and such that for any $\varepsilon > 0$ and any $y \in Y$ there exists a neighborhood U of y satisfying: for any $y_1, y_2 \in U \cap D$ there is an open neighborhood W_1 of y_1 such that each continuous function from a zero-set (in X) into $W_1 \cup \{y_2\}$ admits an extension $f \in \mathscr{C}(X, Y)$ with $\operatorname{diam}(f(X)) < \varepsilon$.

Then (X, Y) satisfies the property $(\mathscr{E})$.

Proof. Choose $y_0 \in Y_1 \cap D$ and set $f_0(x) = y_0$ for all $x \in X$. For any zero-set $F \subset X$ define

$$\Phi(F) = \{f \in \mathscr{C}(X, Y); f|_F \text{ is a constant from } D\}.$$

Clearly $\Phi(F_1) \subset \Phi(F_2)$ whenever $F_1 \supset F_2$.

Let F_1, F_2 be two disjoint zero-sets in X and $V \subset Y$ be open. Choose an arbitrary $y_1 \in V \cap Y_1 \cap D$ and find $f \in \mathscr{C}(X, Y)$ such that $f(F_1) = \{y_1\}$ and $f(F_2) = \{y_0\}$. Then $f \in \Phi(F_1)$, $f(F_1) \subset V$ and $f|_{F_2} = f_0|_{F_2}$, so (ii) from Definition 3.1 is satisfied.

Let us prove (iii) from Definition 3.1. Let $y \in Y$ and $\varepsilon > 0$ be given. Take the neighborhood U of y from (A). Suppose F_1, F_2 are two disjoint zero-sets in X, $f \in \Phi(F_1)$, $f(F_1) \subset U$, $V \subset U$ is open. Let $y_1 \in D \cap U$ be such that $f(F_1) = \{y_1\}$. Choose any $y_2 \in D \cap V$. Let W_1 be the neighborhood of y_1 from (A). It is possible to suppose $W_1 \subset U$. The set $G = f^{-1}(W_1) \setminus F_2$ contains F_1. Let $\varphi \in \mathscr{C}(X, [0, 1])$ be such that $\varphi(F_1) = \{1\}$ and $\varphi(X \setminus G) = \{0\}$. Set $Z = \varphi^{-1}(1/2)$. Then the function $g_1 \in \mathscr{C}(F_1 \cup Z, W_1 \cup \{y_2\})$, defined by

$$g_1(x) = \begin{cases} y_2 & \text{for} \quad x \in F_1, \\ f(x) & \text{for} \quad x \in Z, \end{cases}$$

has an extension $\bar{g}_1 \in \mathscr{C}(X, Y)$ with $\operatorname{diam}(\bar{g}_1(X)) < \varepsilon$. Define

$$g(x) = \begin{cases} \bar{g}_1(x) & \text{for} \quad x \in \varphi^{-1}([1/2, 1]), \\ f(x) & \text{for} \quad x \in \varphi^{-1}([0, 1/2]). \end{cases}$$

Clearly $g \in \mathscr{C}(X, Y)$, and for any $x \in \varphi^{-1}([1/2, 1])$ we have $d(f(x), g(x)) \leqq \leqq d(f(x), y_2) + d(y_2, \bar{g}_1(x)) \leqq \operatorname{diam}(U) + \operatorname{diam}(\bar{g}_1(X)) < 2\varepsilon$ (note that $f(x) \in \in W_1 \subset U$, and $\operatorname{diam}(U) \leqq \varepsilon$ by (A)). So we have found $g \in \Phi(F_1)$ with $g(F_1) \subset V$, $g|_{F_2} = f|_{F_2}$ and $d(f(x), g(x)) < 2\varepsilon$ for all $x \in X$. □

It is easy to see that a metric space Y is locally arcwise connected iff for each $y \in Y$ and $\varepsilon > 0$ there is $\delta > 0$ such that y, z can be joined with an arc of dameter less than ε whenever $d(y, z) < \delta$. This motivates the following definition.

3.5 Definition. A metric space Y is said to be *uniformly locally arcwise connected* if for each $\varepsilon > 0$ there is $\delta > 0$ such that if $y_1, y_2 \in Y$, $d(y_1, y_2) < \delta$ then y_1, y_2 can be joined with an arc of diameter less than ε.

3.6 Definition. Let X be a topological space and Y be a metric space. We shall say that

(a) Y satisfies the *$\mathscr{Z}$-extension property for X* if any continuous function from a zero-set (in X) into Y has an extension from $\mathscr{C}(X, Y)$.

(b) Y satisfies the *local $\mathscr{Z}$-extension property for X* if for each $\varepsilon > 0$ and $y \in Y$ there is a neighborhood U of y such that any continuous function from a zero-set (in X) into U admits an extension $f \in \mathscr{C}(X, Y)$ with $\operatorname{diam}(f(X)) < \varepsilon$.

(c) Y satisfies the *uniform local $\mathscr{Z}$-extension property for X* if for each $\varepsilon > 0$ there is $\delta > 0$ such that any continuous function f from a zero-set $F \subset X$ into Y with $\operatorname{diam}(f(F)) < \delta$ admits an extension $\bar{f} \in \mathscr{C}(X, Y)$ with $\operatorname{diam}(\bar{f}(X)) < \varepsilon$.

The following theorem is a direct consequence of Theorem 3.3 and Theorem 3.4.

3.7 Theorem. Let X be normal and Y metric. Then $\mathscr{B}_1(X, Y) = \mathscr{F}_\sigma(X, Y) \cap \cap \Sigma^*(X, Y)$ provided at least one of the following conditions is satisfied.

(i) Y is arcwise connected and locally arcwise connected.

(i') Y satisfies the $\mathscr{Z}$-extension property for X and the local $\mathscr{Z}$-extension property for X.

(ii) Y contains a dense subspace Y_1 such that Y_1 is arcwise connected and uniformly locally arcwise connected (in the metric generated by that of Y).

(ii') Y contains a dense subspace Y_1 such that Y_1 satisfies the $\mathscr{Z}$-extension property for X and the uniform local $\mathscr{Z}$-extension property for X.

(iii) Y contains a dense subspace Y_1 such that all open balls in Y_1 are arcwise connected.

(iii') Y contains a dense subspace Y_1 such that all open balls in Y_1 satisfy the $\mathscr{Z}$-extension property for X.

3.8 Remark. (a) It is easy to see that all the results (I)–(X) from Introduction follow from Theorem 3.7(i), (i').

(b) The known results (I)–(X) do not cover, for example, the case of $X = [0, 1]$ and Y such that Y is not arcwise connected, $Y_1 \subset Y \subset \mathbb{R}^n$ where $Y_1 = \{y \in \mathbb{R}^n;$ at least one of the coordinates of y is rational$\}$. However, Theorem 3.7(iii) implies $\mathscr{B}_1([0, 1], Y) = \mathscr{F}_\sigma([0, 1], Y)$ (all functions into Y are strongly σ-discrete since Y is separable).

(c) It is not possible to omit the word „uniformly" in Theorem 3.7 (ii), (ii'). Consider $X = [0, 1]$, $Y_1 = \{(t, \sin(1/t)); t > 0\} \subset \mathbb{R}^2$, $Y = Y_1 \cup (\{0\} \times [-1, 1])$. Then Y_1 is a dense arcwise connected and locally arcwise connected subspace of Y. (Hence it satisfies the $\mathscr{Z}$-extension and the local $\mathscr{Z}$-extension property for $[0, 1]$, too.) Since Y is separable, $\Sigma^*(X, Y)$ contains all functions from X into Y. By Theorem $\mathscr{F}$ (and Proposition 1.10) $\mathscr{B}_1(X, Y) \subsetneqq \mathscr{F}_\sigma(X, Y)$, because Y is complete and connected but not locally connected. Moreover, by Theorem 3.7(i), $\mathscr{B}_1(X, Y_1) = \mathscr{F}_\sigma(X, Y_1)$.

(d) Theorem 3.7(i) implies that it is possible to write „normal" instead of „metric" in Theorem $\mathscr{F}$, (d).

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