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On Generalized Hausdorff Semicontinuity of Multifunctions

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The paper deals with a certain characterization of the generalized Hausdorff semicontinuity of multifunctions. The characterization is analogous to the one given in [2]. Since there are many types of generalizations of the continuity it seems to be useful to use such a type which includes the other ones. It was done in [3], where the so called $\mathcal{S}$-continuity is introduced with the help of the concept of a local sieve. The same procedure is followed in this paper.

In the second part some consequences of the characterization for a quasicontinuity are given. If not specified, X denotes a general topological space.

1. Hausdorff $\mathcal{S}$-semicontinuity

In [3] the following concept is introduced.

Definition 1. A family $\mathcal{S}_x$ of subsets of X is called a local sieve at a point $x \in X$ if:

1. $x \in A$ for any $A \in \mathcal{S}_x$
2. $A \subset B$ and $A \in \mathcal{S}_x$ imply $B \in \mathcal{S}_x$
3. $\mathcal{U}_x \subset \mathcal{S}_x$, where $\mathcal{U}_x$ denotes the system of all neighbourhoods of a point x.

If substituting the system of the neighbourhoods $\mathcal{U}_x$ by a local sieve $\mathcal{S}_x$ in the definition of a continuity the so called $\mathcal{S}$-continuity is obtained as it was done in [3]. We shall proceed in a similar way. Let (Y, d) be a metric space. For a set $A \subset Y$ denote $B_\varepsilon[A] = \bigcup\{B_\varepsilon[y] : y \in A\}$, where $B_\varepsilon[y] = \{v \in Y : d(y, v) < \varepsilon\}$. By a multifunction $F : X \to Y$ we mean an assignment that assigns to each x in X a nonempty subset $F(x)$ of Y.

Definition 2. A multifunction $F : X \to Y$ is said to be Hausdorff upper (lower) $\mathcal{S}$-semicontinuous at a point $x \in X$ if for any $\varepsilon > 0$ there exists a set $A \in \mathcal{S}_x$ such that $F(z) \subset B_\varepsilon[F(x)]$ $(F(x) \subset B_\varepsilon[F(z)])$ for any $z \in A$. We shall use the notation H.u.$\mathcal{S}$-s.c. (H.l.$\mathcal{S}$-s.c.).

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Definition 3. Let Z be a topological vector lattice. A function $f : X \to Z$ is said to be lattice upper (lower) $\mathscr{S}$-semicontinuous at a point $x \in X$ if for any neighbourhood V of $f(x)$ there exists a set $A \in \mathscr{S}_x$ such that $f(x) \vee f(u) \in V(f(x) \wedge f(u) \in V)$ for any $u \in A$. We shall use the notation l.u.$\mathscr{S}$-s.c. (l.l.$\mathscr{S}$-s.c.).

By a topological vector lattice we mean a topological vector space equipped with a partial order and two lattice operations. For a precise definition see [1].

The corresponding notions of H.u.$\mathscr{S}$-s.c., H.l.$\mathscr{S}$-s.c., l.u.$\mathscr{S}$-s.c. and l.l.$\mathscr{S}$-s.c. on X are understood as H.u.$\mathscr{S}$-s.c., H.l.$\mathscr{S}$-s.c., l.u.$\mathscr{S}$-s.c. and l.l.$\mathscr{S}$-s.c. at any $x \in X$ respectively. By means of a special selection of a local sieve $\mathscr{S}_x$ we can obtain several types of a generalized continuity.

The set of H.$\mathscr{S}$-semicontinuous multifunctions does not change if the metric d is substituted by a metric ϱ defining the same uniformity on Y. The proof of this fact is similar to the proof of the fact that these two metrics give the same Hausdorff metric on the space of nonempty closed subsets of Y (see [2]). Thus we can replace d by $d_1 = \min(d, 1)$ or in a different way from this moment on we suppose d to be bounded. Further let us suppose F to be closed-valued from this moment on.

Before approaching the characterization of the H,$\mathscr{S}$-semicontinuity we introduce the following notation. For any $x \in X$ and $y \in Y$ denote

$$d(y, F(x)) = \inf\{d(y, z) : z \in F(x)\} .$$

Then we can assign a function $f^* : X \to C(Y)$ to any multifunction $F : X \to Y$. Here $C(Y)$ is a topological vector lattice of bounted continuous real-valued functions on Y considered with the norm of uniform convergence. The value of f^* at a point x is defined as follows

$$f^*(x) : Y \to R$$

$$Y \to d(y, F(x)) .$$

Sometimes we shall write $[f^*(x)](y) = d(y, F(x))$.

This function was introduced in [1], where it was used for the characterization of the Hausdorff semicontinuity. The analogous characterization for the Hausdorff $\mathscr{S}$-semicontinuity is valid. Since the proofs are based on the same ideas, we give only a sketch of the proof for the sake of a completeness.

Theorem 1. *A multifunction $F : X \to Y$ is H.u.$\mathscr{S}$-s.c. (H.l.$\mathscr{S}$-s.c.) at $x \in X$ if and only if a function $f^* : X \to C(Y)$ is l.l.$\mathscr{S}$-s.c. (l.u.$\mathscr{S}$-s.c.) at x.*

Proof. Let $\varepsilon > 0$. Then the equivalence of the following sequence of statements is evident:

1. F is H.u.$\mathscr{S}$-s.c. at x.
2. There exists $A \in \mathscr{S}_x$ such that $F(z) \subset B_\varepsilon[F(x)]$ for any $z \in A$
3. There exists $A \in \mathscr{S}_x$ such that $d(y, F(z)) + \varepsilon > d(y, F(x))$ for any $z \in A$, $y \in Y$.
4. There exists $A \in \mathscr{S}_x$ such that $[f^*(z)](y) + \varepsilon > [f^*(x)](y)$ for any $z \in A$, $y \in Y$.

5. There exists $A \in \mathscr{S}_x$ such that $f^*(z) \wedge f^*(x) \in B_\varepsilon^\infty[f^*(x)]$ for any $z \in A$.

6. f^* is l.l.$\mathscr{S}$-s.c. at x.

Here $B_\varepsilon^\infty[f^*(x)]$ denotes the ε-ball with the centre $f^*(x)$ in the supremum metric on $C(Y)$.

The proof of a dual part is similar. □

Since the values of a function $[f^*(x)]$ are continuous functions from the space Y to R it is evident that $[f^*(x)]$ is determined uniquely by its values on an arbitrary dense subset E of Y. Thus it is sufficient to use only this dense set E in the characterization of a generalized continuity and the following version of Theorem 1 is valid.

Theorem 1a. *Let E be a dense subset of Y. A multifunction $F : X \to Y$ is H.u.$\mathscr{S}$-s.c. (H.l.$\mathscr{S}$-s.c.) at $x \in X$ if and only if a function $f_E^*: X \to C(E)$ is l.l.$\mathscr{S}$-s.c. (l.u.$\mathscr{S}$-s.c.) at x, where $[f_E^*(x)](y) = [f^*(x)](y)$ for any $y \in E$.*

It is well known (see [2]) that there is a natural embedding of the space $(Cl(Y), H_d)$ into the space $(C(Y), d_\infty)$. Here $(Cl(Y), H_d)$ is a space of nonempty closed subsets of X with a Hausdorff metric H_d derived from d. The image of a set $A \in Cl(Y)$ in this embedding is the distance function $d(., A) : Y \to R$. Because of this correspondence it is evident that any type of convergence of the net of multifunctions can be characterized in the following manner:

$$F_\gamma \overset{H_d}{\dashrightarrow} F \quad \text{if and only if} \quad f_\gamma^* \overset{d_\infty}{\dashrightarrow} f^* .$$

Here the symbol $\dashrightarrow$ can be substituted by any type of convergence, which depends only on H_d as for instance pointwise, uniform or transfinite convergences.

2. Quasicontinuity

As already mentioned above the notion of $\mathscr{S}$-continuity includes several types a of generalized continuity, which can be obtained by means of special selections of a local sieve $\mathscr{S}_x$. If we consider local sieve $\mathscr{S}_x$ of the form

$$\mathscr{S}_x = \{x \in X : x \in A, x \in \mathrm{Cl}(\mathrm{Int}\, A)\}$$

we obtain so called quaiscontinuity, which was introduced by Kempisty in 1932. A function $f : X \to Y$ is said to be quasicontinuous at a point $x \in X$ if for any open U containing x and any open V containing $f(x)$ there exists a nonempty open set $G \subset U$ such that $f(z) \in V$ for any $z \in G$. The definitions of such concepts as a Hausdorff semiquasicontinuity or lattice semiquasicontinuity with the help of $\mathscr{S}_x$ are evident. The use of notations H.u.q.c., H.l.q.c., l.u.q.c. and l.l.q.c. is obvious. Here we give some results concerning quasicontinuity, which are direct consequences of the previous characterization.

At first the following notation is needed. Denote the set of all the points at which a multifunction F is not H.u.$\mathscr{S}$-s.c. (H.l.$\mathscr{S}$-s.c.) by $D_{\mathscr{S}}^+F$ ($D_{\mathscr{S}}^-F$). Analogously, the notation $D^+_{\mathscr{S}}f^*$ $(D^-_{\mathscr{S}}f^*)$ for the function f^* and $D^+_{\mathscr{S}}f^*_E$ $(D^-_{\mathscr{S}}f^*_E)$ for the function f^*_E is used.

With respect to the previous section we can see that

$$D^+_{\mathscr{S}}F = D^-_{\mathscr{S}}f^* = D^-_{\mathscr{S}}f^*_E \quad \text{and} \quad D^-_{\mathscr{S}}F = D^+_{\mathscr{S}}f^* = D^+_{\mathscr{S}}f^*_E \tag{1}$$

If the index $\mathscr{S}$ is omitted we obtain equalities concerning usual Hausdorff and lattice semicontinuities.

Proposition 1. *Let Y be a separable metric space and let a multifunction $F : X \to \to Y$ be a pointwise limit of a sequence $\{F_n\}_{n=1}^{\infty}$ of Hausdorff upper (lower) semiquasicontinuous multifunctions. Then D^-F (D^+F) is of the first category.*

Proof. Let E be a countable dense subset of a space Y. According to the equalities (1) and the previous remarks concerning the pointwise convergence, it us sufficient to prove that the set $D^+f^*_E(D^-f^*_E)$ is of the first category. A similar result for the usual semicontinuity of real-valued functions, can be found in [5]. The proof for the lattice semicontinuity is only slightly more complicated, provided that E is countable. So we can omit it.

Instead of that we show the way of using the characterization from Theorem 1.a. Suppose F_n to be H.u.q.c. for any $n \geqq 1$. Then according to Th.1a all the functions $(f^*_n)_E$ are l.l.q.c. and according to [5] D^+f_E is of the first category. From (1) it follows that D^-F is of the first category, too.

The dual result for D^+F can be proved in the same way. □

The second note concerns a transfinite convergence.

Definition 4. A transfinite sequence $\{z_\alpha\}_{\alpha<\Omega}$ (Ω is the first uncountable ordinal number) of elements of a topological space Z is said to be convergent to a point $z \in Z$ if for any neighbourhood V of z there exists $\alpha_0 < \Omega$ such that $z_\alpha \in V$ for any $\alpha \geqq \alpha_0$. The notation $\lim_{\alpha<\Omega} z_\alpha = z$ is used in this case.

Since the topological vector lattice $C(Y)$ with metric d_∞ will be used again, the following consequence of Sierpinski lemma may be stated here (see [6]).

Lemma 1. *Let (f_α) be a transfinite sequence of functions defined on X and taking values in a metric space Z.*

Let $\lim_{\alpha<\Omega} f_\alpha(x) = f(x)$ for any $x \in X$. Let $H \subset X$ be any countable set. Then there exists $\alpha_0 < \Omega$ such that $f_\alpha(x) = f(x)$ for any $x \in H$ and $\alpha \geqq \alpha_0$.

Proposition 2. *Let X be a locally separable and first countable topological space. If $F : X \to Y$ is a pointwise limit of a transfinite sequence of Hausdorff upper (lower) semiquasicontinuous multifunctions, then F is H.u.q.c. (H.l.q.c.).*

Proof. Suppose all F_α to be H.u.q.c. As in the previous case it is sufficient to prove the following implication:

$$f^*_\alpha \text{ are l.l.q.c.} \quad \text{implies} \quad f^* \text{ is l.l.q.c.},$$

where $f^*(x) = \lim_{\alpha<\Omega} f_\alpha^*(x)$ and at the same time f^* is a function corresponding to the multifunction F. An analogous result for the quasicontinuity in [9] is proved and its generalization for a semiquasicontinuity and a lattice semiquasicontinuity does not differ basicaly. So we give only the basic ideas in order to illustrate how the lattice semicontinuity works.

Suppose f^* not to be l.l.q.c. at x. Then an open set U containing x and an open set V containing $f^*(x)$ exist such that for any nonempty open set $G \subset U$ there is $z \in G$ such that $f^*(x) \wedge f^*(z) \notin V$. The neighbourhood U may be supposed to be separable and thus we obtain a countable dense subset H of U such that $f^*(x) \wedge f^*(z) \notin V$ for any $z \in H$. Applying Lemma 1 to the set $H \cup \{x\}$ we obtain

$$f_\alpha^*(z) = f^*(z) \quad \text{for any} \quad z \in H \cup \{x\} \quad \text{and any} \quad \alpha \geqq \alpha_0$$

for suitable $\alpha_0 < \Omega$. Now a lattice lower semiquasicontinuity of f_α^* at x and a density of the set H in U provide a contradiction.

The result concerning H.l.q.c. can be proved in the same way. □

The notion of the quasicontinuity was first introduced in a connection with a separate and joint continuities of a function of two variables. Our last note concerns also this area. Let us remark the same result being obtained directly in a more general case in [8].

Proposition 3. *Let X be a Baire space, Z be a locally second countable. Y be a matric space and a multifunction F be from $X \times Z$ to Y. If all the sections F^x, F^z are H.u.q.c. and all the sections F^2 are H.l.q.c., then F is H.u.q.c.*

Proof: It is easy to see that $(f^x)^* = (f^*)^x$, where $(f^x)^*$ is a function corresponding to the section F^x *and* $(f^*)^x$ is a section of the function f^* corresponding to the multifunction F. Thus it is sufficient to prove a corresponding result concerning joint and separate lattice semiquasicontinuities for the function $f^* : X \times Z \to C(Y)$. Demanded result is really valid in the following form:

Let X, Z be as above. Y be a topological vector lattice and $f^* : X \times Z \to Y$. If all the sections $(f^*)^x$, $(f^*)^z$ are l.l.q.c. and all the sections $(f^*)^z$ are l.u.q.c., then f^* is l.l.q.c..

The proof of this assertion is similar to the proof of the analogous result for the semiquasicontinuity of real-valued functions of two variables. The proof of this result can be found in [4] and so we can omit it. □

It is evident again how to formulate and how to prove the dual result.

Notice that the other characterization of the $\mathscr{S}$-semicontinuity of multifunctions can be found in [7], where also further consequences concerning a general $\mathscr{S}$-continuity are given. Some of them can be also proved by means of Theorem 1. We choose the results introduced here since they are valid for the quasicontinuity only and they do not occur in [7].

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